Geometric conditions for the controllability of parabolic systems

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Control of PDEs
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Outline

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1. Introduction
2. Distributed controllability with space-dependent coefficients in 1D
3. Boundary controllability on a rectangular domain
4. Comments

Controllability of a single heat equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset. It is by now well-known that the heat equation

$$\begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T = (0,T) \times \Omega, \\ y = 0 & \text{on } \Sigma_T = (0,T) \times \partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

is null-controllable in any time $T > 0$ and for any open subset $\omega \subset \Omega$.

Some important references

1. H.O. Fattorini and D.L. Russell (1971).
2. D.L. Russell (1973).
3. G. Lebeau and L. Robbiano (1995).
4. A. Fursikov and O. Yu. Imanuvilov (1996).

The proofs use : Method of moments, transmutation techniques, Carleman estimates.

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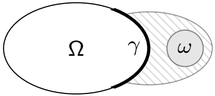
Note that, the boundary controllability is a simple corollary :

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } Q_T, \\ y = 1_\gamma v & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$



Recent results and difficulties for systems

Recently, these ideas have been successfully developped in the framework of systems :

- L. de Teresa (2000), F. Ammar-Khodja *et al.* (2006), S. Guerrero (2007), M. González-Burgos and L. de Teresa (2010).
- E. Fernández-Cara *et al.* (2010), F. Ammar-Khodja *et al.* (2013).
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Though, there are still a lot of open problems. Noteworthy restrictions for systems are :

- **Carleman estimates :** boundary controllability or distributed controllability on a control domain that does not intersect the support of the coupling.
- **Method of moments :** controllability in dimension $N > 1$.
- **Transmutation techniques** : results for wave systems, geometric conditions.

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- **Transmutation techniques** : results for wave systems, geometric conditions.

Not only technical, unexpected behaviors for simple parabolic systems :

1. Distributed and boundary controllability are not equivalent properties.
   E. Fernández-Cara *et al.* (2010).
2. There may be a minimal time of control.
   F. Ammar-Khodja *et al.* (2013).
3. There may be some geometric conditions, as we shall see here.
   G. O. (2013).

Note that these complex situations appear in dimension 1.

We are close from the behaviors of hyperbolic equations (C. Bardos *et al.* (1992)) or parabolic degenerate equations (K. Beauchard *et al.* (2011)).

In this talk, we consider the following parabolic systems of n equations :

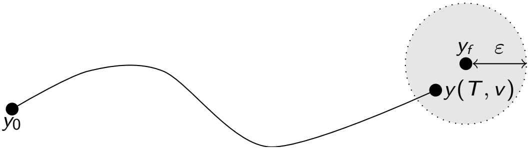
$$\left\{\begin{array}{ll} \partial_t y - \Delta y = A(x)y + 1_\omega Bv & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{ll} \partial_t y - \Delta y = A(x)y & \text{in } Q_T, \\ y = 1_\gamma Bv & \text{on } \Sigma_T, \\ y(0) = y_0 & \text{in } \Omega. \end{array}\right. \qquad (\mathcal{S})$$

(distributed controllability) (boundary controllability)

- $y = (y_1, \ldots, y_n)$ is the state and y_0 the initial data.
- $A \in L^\infty(\Omega; \mathcal{M}_n(\mathbb{R}))$ links the equations.
- $v \in L^2(Q_T)$ (or $v \in L^2(\Sigma_T)$) is the control : only one control for $n > 1$ equations.
- $B \in \mathbb{R}^n$ situates **algebraically** the control.
- $\omega \subset \Omega$ (or $\gamma \subset \partial\Omega$) situates **geometrically** the control.

Approximate controllability and observability

In this talk, we mainly focus on the approximate controllability of $(\mathcal{S})$ (weaker than the null-controllability).



Let us recall that the system $(\mathcal{S})$ is approximately controllable if and only if its adjoint system

$$\begin{cases} -\partial_t z - \Delta z = A^* z & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(T) = z_f & \text{in } \Omega, \end{cases} \qquad (\mathcal{S}^*)$$

satisfies the following unique continuation property (for instance for the distributed controllability) :

$$\forall z_f \in L^2(\Omega)^n, \quad \left(1_\omega B^* z(t) = 0, \quad \forall t \in (0, T) \right) \Longrightarrow z_f = 0.$$

Actually, we have an easier characterization for the approximate controllability :

Theorem (H.O. Fattorini (1966))

Under appropriate assumptions (parabolic) on the operators $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \longrightarrow H$ and $\mathcal{C} : \mathcal{D}(\mathcal{C}) \subset H \longrightarrow U$, the unique continuation property

$$\forall z_f \in \mathcal{D}(\mathcal{A}^*), \quad \Big(\mathcal{C}^* z(t) = 0 \text{ a.e. } t \in (0, +\infty)\Big) \Longrightarrow z_f = 0,$$

where z solves the adjoint system with z_f, is equivalent to

$$\ker(s - \mathcal{A}^*) \cap \ker \mathcal{C}^* = \{0\}, \quad \forall s \in \mathbb{C}.$$

1. In finite dimension, this gives an equivlalent characterization to the so-called Kalman rank condition :
$$\ker(s - A^*) \cap \ker B^* = \{0\}, \quad \forall s \in \mathbb{C}.$$
2. Looking at the boundary approximate controllability, this writes
$$\forall \phi \in \ker(s - \Delta), \quad \partial_n \phi = 0 \text{ on } \gamma \Longrightarrow \phi = 0, \quad \forall s \in \mathbb{C}.$$

R. C. MacCamy *et al.* (1968).

Outline

Structural assumptions

We consider several possible structures of $A(x)$ and discuss the distributed controllability :

1. Controllability of a 2×2 cascade system

$$A(x) = \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2. Simultaneous controllability of several 2×2 cascade systems

$$A(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ a_{31}(x) & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

3. Controllability of a 3×3 cascade system

$$A(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

They are toy models and they have a good spectral structure.

State of the art

We denote $\mathcal{O}_2 = \operatorname{supp}(a_{21})$ and $\mathcal{O}_3 = \operatorname{supp}(a_{32})$.

References

Problems which are completely solved :

- $A(x) = A$ is constant.
  F. Ammar-Khodja *et al.* (2009).
- Case $\mathcal{O}_2 \cap \mathcal{O}_3 \cap \omega \neq \emptyset$.
  M. González-Burgos and L. de Teresa (2010).

Problems with partial answers :

- Case $\mathcal{O}_2 \cap \omega = \emptyset$, a sufficient condition of approximate controllability.
  O. Kavian and L. de Teresa (2010).
- Case $\mathcal{O}_2 \cap \omega = \emptyset$, null-controllability holds for positive coupling terms and under Geometric Control Condition.
  L. Rosier and L. de Teresa (2011), F. Alabau-Boussouira (2012).

All these results are actually **true in any dimension N.**

Reduction to a non-homogeneous scalar elliptic problem

According to the theorem of Fattorini we have to investigate the following property :

$$\left.\begin{array}{rl} -\Delta u - A(x)^* u = su & \text{in } \Omega \\ u_1 = 0 & \text{in } \omega \end{array}\right\} \Longrightarrow u_1 = \ldots = u_n = 0 \text{ in } \Omega, \quad \forall u \in \mathcal{D}(-\Delta)^n, \forall s \in \mathbb{C}.$$

For instance, for the simultaneous controllability of two 2×2 cascade systems

$$A(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ a_{31}(x) & 0 & 0 \end{pmatrix},$$

the task is reduced to prove

$$\left.\begin{array}{rl} -\Delta u_1 - su_1 = a_{21} u_2 + a_{31} u_3 & \text{in } \Omega, \\ -\Delta u_2 = su_2 & \text{in } \Omega, \\ -\Delta u_3 = su_3 & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega, \end{array}\right\} \Longrightarrow u_1 = u_2 = u_3 = 0, \quad \forall u_1, u_2, u_3 \in \mathcal{D}(-\Delta).$$

To deal with this kind of system we will study the UCP for the non-homogeneous scalar elliptic equation :

$$-\Delta u_1 - su_1 = F \quad \text{in } \Omega.$$

Settings

- From now on, we will restrain our study to the 1D case and take $\Omega = (0,1)$. $\omega \subset \Omega$ still denotes the control domain. ω does not need to be connected !
- Let ϕ_k be the Dirichlet eigenfunctions of $-\partial_x^2$ and λ_k the corresponding eigenvalues.
- We denote by $\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)$ the set of all connected components of $\overline{\Omega\backslash\omega}$.
- For every $C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right)$ and $F \in L^2(\Omega)$, we define the vector $M_k(F,C) \in \mathbb{R}^2$ by

$$M_k(F,C) = \begin{pmatrix} \int_C F\phi_k\, dx \\ 0 \end{pmatrix} \text{ if } C \cap \partial\Omega \neq \emptyset, \quad M_k(F,C) = \begin{pmatrix} \int_C F\phi_k\, dx \\ \int_C F\phi_k'\, dx \end{pmatrix} \text{ if } C \cap \partial\Omega = \emptyset,$$

Example :

$$\Longrightarrow M_k(F,C) = \begin{pmatrix} \int_C F\phi_k\, dx \\ 0 \end{pmatrix}, \quad \forall C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right).$$

- Then, for any $F \in L^2(\Omega)$ we define the following family of vectors of $\mathbb{R}^2$

$$\mathcal{M}_k(F,\omega) = (M_k(F,C))_{C\in\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)} \in (\mathbb{R}^2)^{\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)}.$$

Theorem (F. Boyer and G.O. (2013))

There exists a solution $u \in \mathcal{D}\left(-\partial_x^2\right)$ to the following overdetermined problem

$$\begin{cases} -\partial_x^2 u - \lambda_k u = F & \text{in } \Omega, \\ u = 0 & \text{in } \omega, \end{cases}$$

if and only if

$$\begin{cases} F = 0 & \text{in } \omega, \\ \mathcal{M}_k(F,\omega) = 0. \end{cases}$$

Application - simultaneous controllability

We consider

$$A(x)=\begin{pmatrix} 0 & \cdots & \cdots & 0\\ a_{21}(x) & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}(x) & 0 & \cdots & 0\end{pmatrix},\quad B=\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}.$$

In view of the approximate controllability we can always assume that

$$a_{i1}1_\omega=0,\quad \forall i\in\{2,...,n\}.$$

Theorem (F. Boyer and G. O. (2013))

Then, the system

$$\begin{cases}\partial_t y-\partial_x^2 y=A(x)y+1_\omega Bv & \text{in } Q_T,\\ y=0 & \text{in } \Sigma_T,\end{cases}$$

is approximately controllable if and only if

$$\forall k\geq 1,\quad \operatorname{rank}\ \{\mathcal{M}_k(a_{21}\phi_k,\omega),\dots,\mathcal{M}_k(a_{n1}\phi_k,\omega)\}=n-1.$$

Minimal number of connected components to control :

$$2\,\mathrm{card}\,\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)\geq n-1.$$

A system of $n=6$ equations when ω is an interval is NOT approximately controllable.

Proof "$\Longrightarrow$"

There exists $(\delta_2, \dots, \delta_n) \neq (0, \dots, 0)$ such that

$$\sum_{i=2}^{n} \delta_i \mathcal{M}_k\left(a_{i1}\phi_k, \omega\right) = \mathcal{M}_k\left(\sum_{i=2}^{n} \delta_i a_{i1}\phi_k, \omega\right) = 0.$$

Denoting $F = \sum_{i=2}^{n} \delta_i a_{i1}\phi_k$ we have $F = 0$ in ω and thus there exists u_1 such that

$$\begin{cases} -\partial_x^2 u_1 - \lambda_k u_1 = F & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega. \end{cases}$$

As a result,

$$u = \begin{pmatrix} u_1 \\ \delta_2\phi_k \\ \vdots \\ \delta_n\phi_k \end{pmatrix}$$

solves

$$\begin{cases} -\partial_x^2 u - \lambda_k u = A(x)^* u & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega, \\ u \not\equiv 0, & \end{cases}$$

and the approximate controllability fails. □

Simple conditions for a 2×2 system

We consider the 2×2 system

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 = 1_\omega v & \text{in } Q_T, \\ \partial_t y_2 - \partial_x^2 y_2 = a_{21}(x) y_1 & \text{in } Q_T. \end{cases} \tag{1}$$

Corollary (F. Boyer and G. O. (2013))

Let $\mathcal{O}_2 = \operatorname{supp}(a_{21})$.

1. *Case* $\mathcal{O}_2 \cap \omega \neq \emptyset$ *: System (1) is approximately controllable.*
2. *Case* $\mathcal{O}_2 \cap \omega = \emptyset$ *:*
   a) ***Sufficient condition :*** *System (1) is approximately controllable if* a_{21} *satisfies*

$$\int_\Omega a_{21} (\phi_k)^2 \, dx \neq 0, \quad \forall k \geq 1. \tag{2}$$

   b) ***Necessary condition :*** *if System (1) is approximately controllable and* $\omega, \mathcal{O}_2$ *are connected, then (2) has to hold.*

Work in progress for the null-controllability.
F. Ammar-Khodja *et al.*

Remarks

- The condition

$$\int_{\Omega} a_{21}(\phi_k)^2\, dx \neq 0, \quad \forall k \geq 1. \tag{3}$$

  can be found in O. Kavian and L. de Teresa (2010).
  It is actually a sufficient condition of approximate controllability **in any dimension** $N \geq 1$.
- In general, condition (3) is not necessary if $\omega, \mathcal{O}_2$ are not anymore connected, see right after.
- Condition (3) is easily checkable (1D). Example : $a_{21} \geq 0$ and $a_{21} \not\equiv 0$.
  No result available for the wave systems without such a sign assumption.

Geometry of the control domain

Let us have a look at the 2×2 system

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 = 1_\omega v & \text{in } Q_T, \\ \partial_t y_2 - \partial_x^2 y_2 = a_{21}(x) y_1 & \text{in } Q_T, \end{cases} \tag{4}$$

with

$$a_{21}(x) = \left(x - \frac{1}{2}\right) 1_{\mathcal{O}_2}(x), \quad \mathcal{O}_2 = \left(\frac{1}{4}, \frac{3}{4}\right).$$

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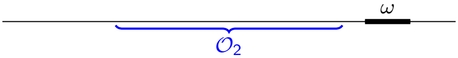
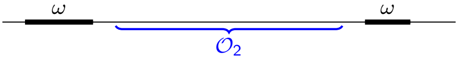
We consider the two following geometric situations for ω :



(c) ω is connected



(d) ω is not connected

Very different behaviors :

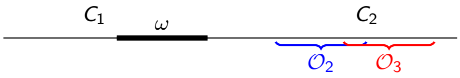
- System (4) is **NOT** approximately controllable in Figure (c).
- System (4) is approximately controllable in Figure (d).

Simultaneous controllability of two 2×2 systems - position of the coupling domains

Let us now consider the system

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 = 1_\omega v & \text{in } Q_T, \\ \partial_t y_2 - \partial_x^2 y_2 = a_{21}(x) y_1 & \text{in } Q_T, \\ \partial_t y_3 - \partial_x^2 y_3 = a_{31}(x) y_1 & \text{in } Q_T. \end{cases} \tag{5}$$

Let $\mathcal{O}_2 = \mathrm{supp}\,(a_{21})$ and $\mathcal{O}_3 = \mathrm{supp}\,(a_{31})$.
Then, in the following configuration



(e) Coupling terms in the same connected component of $\overline{\Omega \setminus \omega}$

System (5) is **NOT** approximately controllable.

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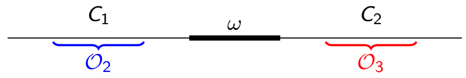

(f) Coupling terms in different connected components of $\overline{\Omega \backslash \omega}$

System (5) is null/approximately controllable IF AND ONLY IF so are the following 2×2 systems

$$\begin{cases} \partial_t \hat{y}_1 - \partial_x^2 \hat{y}_1 = 1_\omega \hat{v} & \text{in } Q_T, \\ \partial_t \hat{y}_2 - \partial_x^2 \hat{y}_2 = a_{21}(x) \hat{y}_1 & \text{in } Q_T, \end{cases} \quad \begin{cases} \partial_t \tilde{y}_1 - \partial_x^2 \tilde{y}_1 = 1_\omega \tilde{v} & \text{in } Q_T, \\ \partial_t \tilde{y}_3 - \partial_x^2 \tilde{y}_3 = a_{31}(x) \tilde{y}_1 & \text{in } Q_T. \end{cases}$$

Observe that we do not require the same control ! ! !

Outline

Some known results

In this part $A(x) = A$ is constant and we discuss the boundary controllability.

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We denote by $\{-\lambda_k\}_k$ and $\{\theta_i\}_i$ the distincts eigenvalues of Δ with Dirichlet boundary condition and A^*. The eigenvalues of the operator $\Delta + A^*$ are then $\{-\lambda_k + \theta_i\}_{k,i}$.

Warning resonances : with this notation we may have, for some indices

$$-\lambda_k + \theta_i = -\lambda_{k'} + \theta_{i'}, \quad \text{with } (k,i) \neq (k',i').$$

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We recall that :

- In dimension $N = 1$ the system

$$\begin{cases} \partial_t y - \partial_{x_1}^2 y = Ay & \text{in } (0,T) \times (0, X_1), \\ y(t,0) = Bv(t), \quad y(t,X_1) = 0 & \text{on } (0,T). \end{cases}$$

  is null/approximately controllable in time $T > 0$ if and only if we have the Kalman rank condition and

$$-\lambda_k + \theta_i \neq -\lambda_{k'} + \theta_{i'}, \quad \text{for all } (k,i) \neq (k',i'). \tag{6}$$

  E. Fernández-Cara *et al.* (2010) ; F. Ammar-Khodja *et al.* (2011).
- In dimension N>1, there exists some partial results under the **Geometric Control Condition on** γ and with a **structural assumption on** A (that avoids the resonances).
  F. Alabau-Boussouira and M. Léautaud (2012) ; F. Alabau-Boussouira (2013).
- The non-resonance condition (6) is a sufficient condition of approximate controllability **in any dimension** N and **for any control domain** $\gamma \subset \partial\Omega$.
  G. O. (2013).

Framework

In what follows we do not assume the non-resonance condition

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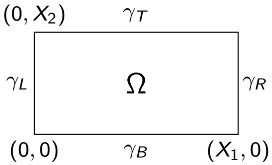
We are going to discuss the controllability properties of the system

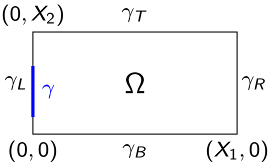
$$\begin{cases} \partial_t y - \Delta y = Ay & \text{in } Q_T, \\ y = 1_\gamma Bv & \text{on } \Sigma_T, \end{cases}$$

for the following particular geometry :



Controllability on one face of the rectangle

We assume that γ is included in only one face of the rectangle :



Theorem (A. Benabdallah *et al.* (2013))

For any $\gamma \subset \gamma_L$, the 2D system

$$\begin{cases} \partial_t y - \Delta y = Ay & \text{in } Q_T, \\ y = 1_\gamma B v & \text{on } \Sigma_T, \end{cases}$$

is null/approximately controllable ***if and only if*** *so is the following 1D system :*

$$\begin{cases} \partial_t y - \partial_{x_1}^2 y = Ay & \text{in } (0,T) \times (0, X_1), \\ y(t,0) = Bv(t), \quad y(t, X_1) = 0 & \text{on } (0,T). \end{cases} \tag{7}$$

Recall that we have a complete characterization of the controllability of (7).
The idea is to project on the dimension 1. H.O. Fattorini (1975).

Some consequences

In particular, we see that :

1. The 2D non-resonance condition is not necessary :

$$-\lambda_k^{(0,\pi)} - \lambda_4^{(0,1)} + 0 = -\lambda_k^{(0,\pi)} - \lambda_5^{(0,1)} + 9\pi^2.$$

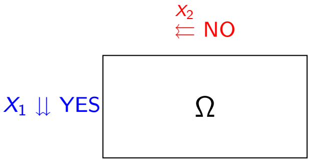
Some consequences

In particular, we see that :

1. The 2D non-resonance condition is not necessary :

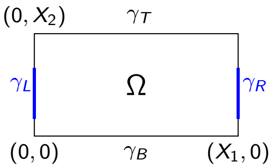
$$-\lambda_k^{(0,\pi)} - \lambda_4^{(0,1)} + 0 = -\lambda_k^{(0,\pi)} - \lambda_5^{(0,1)} + 9\pi^2.$$

2. The controllability of the system depends on where the control is acting :



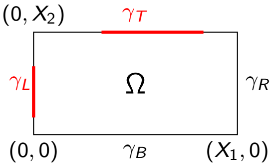
Controllability on two faces of the rectangle

We assume now that γ is included in two faces of the rectangle :



- Case $\gamma \subset \gamma_L \cup \gamma_R$: same situation as previously.

We assume now that γ is included in two faces of the rectangle :



- Case $\gamma \subset \gamma_L \cup \gamma_T$:

Theorem (G. O. (2013))

Under the Kalman rank condition (necessary), if

$$\gamma \cap \gamma_L \neq \emptyset, \quad \gamma \cap \gamma_T \neq \emptyset,$$

then the system of 2 equations

$$\begin{cases} \partial_t y - \Delta y = Ay & \text{in } Q_T, \\ y = 1_\gamma Bv & \text{on } \Sigma_T, \end{cases}$$

is approximately controllable.

Work in progress for the null-controllability.

Outline

Comments

- Connections between the two problems : geometry can creates controls.
- Open problems : dimension $N > 1$, null-controllability, no structural assumption on $A(x)$, other geometries,...

THANK YOU FOR YOUR ATTENTION !