Averaged control

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Leed a Euler, leed a Euler, él es el maestro de todos nosotros.

(Pierre Simon Laplace)
Lisez le Courant-Hilbert....
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2 Finite-dimensional systems
   - Averaged control
   - Averaged observation
   - Comparison with simultaneous controllability

3 Averaged control of PDEs: The finite case
   - Finite averages of wave equations
   - Additive perturbations of PDE
   - Systems of heat equations
   - Ingham like inequalities

4 Averaged control of PDEs: Continuous averages
   - Continuous averages of heat equations
   - An abstract setting
   - Continuous averages of wave equations

5 Perspectives and open problems
Motivation

- Often parameters of the control system under consideration are not fully known.
- It is then natural to look for robust control strategies, independent of the unknown parameters, and performing, overall, optimally.
- We introduce the notion of “averaged control” that consists, simply, on controlling the average of solutions with respect to the unknown parameters.
- As we shall see, this leads to interesting new problems in the frame of observability of parameter dependent systems.
- The problems are linked, but different, to those arising in the context of simultaneous observation and control.
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5 Perspectives and open problems
Consider the finite dimensional linear control system

\[
\begin{align*}
    x'(t) &= A(\nu)x(t) + Bu(t), \quad 0 < t < T, \\
    x(0) &= x^0.
\end{align*}
\]  

(1)

In (1) the (column) vector valued function

\[ x(t, \nu) = (x_1(t, \nu), \ldots, x_N(t, \nu)) \in \mathbb{R}^N \]

is the state of the system, \( A(\nu) \) is a \( N \times N \)-matrix and \( u = u(t) \) is a \( M \)-component control vector in \( \mathbb{R}^M \), \( M \leq N \).

- The matrix \( A \) is assumed to depend on a parameter \( \nu \) in a continuous manner. To fix ideas we will assume that the parameter \( \nu \) ranges within the interval \( (0, 1) \).

- Note however that, to simplify the presentation, the control operator \( B \) has been taken to be independent of \( \nu \), the same as the initial datum \( x_0 \in \mathbb{R}^N \) to be controlled. But the same analysis applies when both of them depend on \( \nu \).
Given a control time $T > 0$ and a final target $x^1 \in \mathbb{R}^N$ we look for a control $u$ such that the solution of (1) satisfies

$$\int_0^1 x(T, \nu) d\nu = x^1. \quad (2)$$

This concept of averaged controllability differs from that of simultaneous controllability in which one is interested on controlling all states simultaneously and not only its average.

When $A$ is independent of the parameter $\nu$, controllable systems can be fully characterized in algebraic terms by the rank condition

$$\text{rank} \left[ B, AB, \ldots, A^{N-1}B \right] = N. \quad (3)$$
The following holds:

**Theorem**

Averaged controllability holds if and only the following rank condition is satisfied:

\[
\text{rank} \left[ B, \int_0^1 [A(\nu)] d\nu B, \int_0^1 [A(\nu)]^2 d\nu B, \ldots \right] = N. \tag{4}
\]
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Let us now characterize the property of averaged control in terms of the observability of the adjoint system.

The adjoint system depends also on the parameter $\nu$:

$$
\begin{cases}
-\psi'(t) = A^*(\nu)\psi(t), & t \in (0, T) \\
\psi(T) = \psi^0.
\end{cases}
$$

Note that, for all values of the parameter $\nu$, we take the same datum for $\psi$ at $t = T$. This is so because our analysis is limited to the problem of averaged controllability.

The corresponding **averaged observability** property reads:

$$
|\psi^0|^2 \leq C \int_0^T \left| B^* \int_0^1 \psi(t, \nu)d\nu \right|^2 dt.
$$
The reason is the following duality identity:

\[
< \int x(T, \nu) d\nu, \varphi^0 > - < x^0, \int \varphi(0, \nu) d\nu > = \int_0^T < u(t), B^* \int \varphi(t, \nu) d\nu >.
\]

In fact, once the averaged observability inequality above is satisfied, the control of minimal \( L^2(0, T) \)-norm can be built by minimizing the quadratic functional below within the class of solutions of the parameter-dependent adjoint system, i.e. minimizing

\[
J(\varphi^0) = \frac{1}{2} \int_0^T \left| \int_0^1 B^*(\nu) \varphi(t, \nu) d\nu \right|^2 dt
- < x^1, \varphi^0 > + < x^0, \int_0^1 \varphi(0, \nu) d\nu >
\]

in \( \mathbb{R}^N \).

The functional is continuous and convex, and its coercivity is guaranteed by the averaged observability inequality. The control is then

\[
u(t) = B^* \int \tilde{\varphi}(t, \nu) d\nu,
\]

where \( \tilde{\varphi} \) is the solution of the adjoint system associated to the minimizer \( \varphi^0 \) of the functional \( J \).
Since we are working in the finite-dimensional context, the observability inequality (6) is equivalent to the following uniqueness property:

$$B^* \int_0^1 \varphi(t, \nu) d\nu = 0 \quad \forall t \in [0, T] \Rightarrow \varphi^0 \equiv 0.$$  \hspace{1cm} (8)

To analyze this inequality we use the following representation of the adjoint state:

$$\varphi(t, \nu) = \exp[A^*(\nu)(T - t)]\varphi^0.$$  

Then, the fact that

$$B^* \int_0^1 \varphi(t, \nu) d\nu = 0 \quad \forall t \in [0, T]$$

is equivalent to

$$B^* \int_0^1 \exp[A^*(\nu)(t - T)] d\nu \varphi^0 = 0 \quad \forall t \in [0, T].$$

The result follows using the time analyticity of the matrix exponentials, and the classical argument consisting in taking consecutive derivatives at time $t = T$. 
Contrarily to the classical rank condition for the controllability of a given system, in the present context of averaged control, all moments, of an arbitrarily high order, need to be taken into account in the characterization.

For instance, if the dependence of $A(\nu)$ with respect to $\nu$ is odd, we see that all the terms involving an odd power vanish, and only the even ones remain...
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5. Perspectives and open problems
There is an extensive literature on the simultaneous control of systems, mainly in the PDE setting. The question is then whether one can control different dynamics by means of the same control function. In the context we are working this would consist on considering the state equation

\[
\begin{align*}
\dot{x}(t) &= A(\nu)x(t) + Bu(t), \quad 0 < t < T, \\
x(0) &= x^0(\nu),
\end{align*}
\]  

(9)

with initial data depending on \( \nu \), and then looking for a control \( u = u(t) \), such that the solution \( x(t, \nu) \) satisfies

\[ x(T, \nu) = x^1, \quad \forall \nu. \]

This, of course, requires much stronger observability inequalities as well.
Finite-dimensional systems  
Comparison with simultaneous controllability

To better clarify the situation we consider the case where $\nu$ takes two possible values, so that the system only has two possible modes, governed by the operators $A_1$ and $A_2$. The parameter dependent system then reads:

$$
\begin{aligned}
\left\{ \begin{array}{l}
x'_j(t) = A_j x_j(t) + Bu(t), \quad 0 < t < T, \\
x_j(0) = x^0_j, \quad j = 1, 2.
\end{array} \right.
\end{aligned}
$$

(10)

Similarly, the adjoint system can be written as:

$$
\begin{aligned}
\left\{ \begin{array}{l}
-\varphi'_j(t) = A^*_j \varphi_j(t), \quad t \in (0, T) \\
\varphi_j(T) = \varphi^0_j, \quad j = 1, 2.
\end{array} \right.
\end{aligned}
$$

(11)

The simultaneous observability inequality reads

$$
|\varphi^0_1|^2 + |\varphi^0_2|^2 \leq C \int_0^T |B^* [\varphi_1 + \varphi_2]|^2 dt, \quad \forall \varphi_j^0 \in \mathbb{R}^N, \quad j = 1, 2.
$$

(12)

For averaged controllability it is sufficient this to hold in the particular case where $\varphi^0_1 = \varphi^0_2$. 

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Summarizing

- **Averaged control:**
  - We control the average of the states.
  - The observation is required only for the adjoint states such that each component departs from the same datum.
  - Minimization is done over that subclass of solutions of the adjoint system.

- **Simultaneous control:**
  - We control each component of the state.
  - The observation is required for all adjoint states.
  - Minimization is done over the whole class of solutions of the adjoint system.

One can connect one problem to the other by means of a continuation/penalization procedure. For instance, averaged control refers to the control of $x_1(T) + x_2(T)$, letting $x_1(T) - x_2(T)$ free. One can link this property to the more demanding one of simultaneous control by means of a penalization of the form $|x_1(T) - x_2(T)| \leq k$, with $k$ ranging in $[0, \infty)$. 
Simulations developed by J. Loheac
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The same analysis can be developed for systems of wave equations depending on parameters\(^2\)

\[
\partial_{tt} u_i - (c_i(t, x) \nabla u_i) = 0, \quad u_j|_\Sigma = 0, \quad u_i(0, \cdot) = \gamma, \quad \partial_t u_i(0, \cdot) = \beta, \quad i = 1, 2.
\]

We assume the coefficients to be \(C^{1,1}\) so that the bicharacteristic rays are well-defined.

Our analysis, based on previous works on the propagation of microlocal defect measures, applies, both for manifolds without boundary or for bounded domains with, say, Dirichlet boundary conditions.

In what concerns the problem of observability, the question can be formulated as follows:

Under which conditions we can get the following observability estimate

$$E(0) \leq C \int_0^T \int_0^\infty |\nabla u_1 + \nabla u_2|^2 dx dt?$$

where

$$E(0) := ||\nabla x \gamma||_{L^2}^2 + ||\beta||_{L^2}^2.$$ 

Note that the main difference with respect to previous works on simultaneous control is that the initial data of both wave equations are taken to be the same.
Theorem

(M. Lazar & E. Z.)

Assume

- \((0, T) \times \omega \subseteq \mathbb{R} \times \Omega\) satisfies GCC for the Wave Equation \# 1.
- \(c_1(t, x) > c_2(t, x) > 0, (t, x) \in (0, T) \times \omega\).

Then there exists a constant \(C\) such that

\[
E(0) \leq C \int_0^T \int_\omega |\nabla u_1 + \nabla u_2|^2 dx dt.
\]

The time of control is that of the fast velocity of propagation.
The proof

- Goes by contradiction and uses microlocal defect or H-measures.
- The different velocities of propagation makes the support of the measures to be disjoint.
- Each measure propagates along the corresponding bicharacteristics.
- The contradiction is reached as soon as one of the measures gets to the initial time.
Some references (non exhaustive...)

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Consider the following more complex system

\[
\begin{align*}
    u_{1,t} - \Delta u_1 &= 0, \quad (t, x) \in \mathbb{R} \times \Omega \\
    u_{2,tt} - \Delta u_2 &= 0, \quad (t, x) \in \mathbb{R} \times \Omega \\
    u_i &= 0, \quad (t, x) \in \mathbb{R} \times \partial \Omega, \quad i = 1, 2 \\
    u_1(0, \cdot) &= \varphi(\cdot) \quad x \in \Omega.
\end{align*}
\]

We assume that **the main dynamics**, the one we want to observe, is given by \( u_1 \), governed by the heat equation. **The wave solution, \( u_2 \), is then adding an additive perturbation.** Note that no information is given on \( u_2 \), other than being the solution of the wave equation. In particular, nothing is known on its initial data.
We know that there exists a constant $C$ such that the following estimate holds

$$
\sum_k e^{-c\sqrt{\lambda_k}} |\hat{\phi}_k|^2 \leq C \int_0^T \int_\omega |u_1|^2 \, dx \, dt.
$$

(13)

We claim that

$$
\sum_k e^{-c\sqrt{\lambda_k}} |\hat{\phi}_k|^2 \leq C \int_0^T \int_\omega |u_1 + u_2|^2 \, dx \, dt,
$$

(14)

for all solutions $(u_1, u_2)$ of the above system.

---

Proof

\[ P_1 = \partial_t - \Delta, \quad P_2 = \partial_{tt} - \Delta. \]

Observe that

\[ v_1 = P_2(u_1 + u_2) = P_2 u_1. \quad (15) \]

solves the heat equation with the same Dirichlet boundary conditions:

\[ v_{1,t} - \Delta v_1 = 0, \quad v_1|\Sigma = 0, \quad v_1(0) = [u_{1,tt} - \Delta u_1](0) = [\Delta^2 - \Delta](\varphi). \]

The proof follows, using

\[ \sum_k e^{-c\sqrt{\lambda_k}} |\hat{\phi}_k|^2 \leq C_s \| v_1 \|^2_{H^{-s}(\omega \times (0, T))}. \quad (16) \]

and

\[ \| v_1 \|_{H^{-2}(\omega \times (0, T))} = \| P_2(u_1) \|_{H^{-2}(\omega \times (0, T))} = \| P_2(u_1 + u_2) \|_{H^{-2}(\omega \times (0, T))} \]

\[ \leq C \| u_1 + u_2 \|_{L^2(\omega \times (0, T))}. \]
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For simplicity we state the result in the case of two equations:

\[ u_{1,t} - \Delta u_1 = 0, \; u_{2,t} - c_2 \Delta u_2 = 0, \]

\[ u_i|_{\Sigma} = 0, \; i = 1, 2, \]

\[ u_1(0, \cdot) = \varphi \in L^2(\Omega). \]

We underline that no information is provided on the initial datum of \( u_2 \).

**Theorem**

Let \( \omega \) an open non-empty subset of \( \Omega \), \( c_2 > 0 \), \( c_2 \neq 1 \) and \( T > 0 \). Then,

\[ \|u_1(T)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\omega |u_1 + u_2|^2 ddt \]  

(17)

for all solutions \((u_1, u_2)\) of the above system.

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4 To be compared with the existing wide literature on the control of systems of heat equations, Stokes equations, often with a limited number of controllers... See lectures by N. Carreño, M. González-Burgos, P. Lissy and related refs.
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The proof we have applied here is inspired in the techniques we employed in ⁵ to deal with wave equations on graphs. There the time-periodicity of solutions of the 1 – d equation was used to annihilate some of the components of the system under consideration, constituted by the deformation of each string of the graph.

Consider two non-harmonic Fourier series, representing the values at a given point of the solutions of the 1 – d wave and heat equations, respectively:

\[ u_h(t) = \sum_k a_k e^{-k^2 t}, \quad u_w(t) = \sum_k b_k e^{ikt}, \]  

(18)

and their superposition

\[ u(t) = u_h(t) + u_w(t) = \sum_k [a_k e^{-k^2 t} + b_k e^{ikt}], \]  

(19)

We claim that for all \( T \geq 2 \) there exist positive constants \( c = c(T), C > 0 \) such that

\[ \sum_k e^{-c(T)k^2} |a_k|^2 \leq C \int_0^T |u_h(t) + u_w(t)|^2 dt, \]  

(20)
Using that $u_w$ is time-periodic of time-period 2, we have
\[ u(t + 2) - u(t) = u_h(t + 2) - u_h(t) = \sum_k [(e^{-2k^2} - 1) a_k e^{-k^2 t}]. \]

Applying well-known inequalities for series of real exponentials \(^6\) we get
\[ \sum_k |e^{-2k^2} - 1|^2 a_k^2 e^{-(T-2)k^2} \leq C \int_0^T |u(t+2) - u(t)|^2 dt \leq C \int_0^T u^2(t) dt, \]
and this easily leads to the result.

Note that, the most striking aspect of the estimate is that, it is totally independent of the Fourier coefficients $\{b_k\}$ governing the wave-like perturbation. It seems that this surprising stability property of these classical inequalities for non-harmonic Fourier series was not noticed before. It would be interesting to make a more systematic analysis of these issues for the interaction of more general wave and heat spectra.

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So far we have analyzed the property of averaged control in, mainly, two situations:

- Continuous averages of finite-dimensional systems;
- Finitely many PDE.

What about continuous averages of PDE?
Let us consider the heat equation, depending on a diffusivity coefficient $\sigma$ ranging on a bounded interval $0 < \sigma_1 \leq \sigma \leq \sigma_2 < \infty$:

$$u_t - \sigma \Delta u = 0, \quad (t, x) \in \mathbb{R} \times \Omega.$$ 

All equations are assumed to take $u^0 \in L^2(\Omega)$ as initial datum, and the same boundary conditions (Dirichlet ones to fix ideas). Define the average

$$v(x, t) = \int_{\sigma_1}^{\sigma_2} u(x, t; \sigma) \, d\sigma. \quad (21)$$

Can we identify the dynamical system governing the evolution of $v$?
Note that
\[ u(x, t; \sigma) = U(x, \sigma t), \]
where \( U \) solves the same heat equation with diffusivity \( \sigma = 1 \). Thus,
\[ v(x, t) = \int_{\sigma_1}^{\sigma_2} u(x, t; \sigma) \, dx = \int_{\sigma_1}^{\sigma_2} U(x, \sigma t) \, d\sigma = \frac{1}{t} [V(x, \sigma_2 t) - V(x, \sigma_1 t)], \]
(22)
where
\[ V(x, t) = \int_0^t U(x, s) \, ds + \xi(x), \]
\[-\Delta \xi = u^0, \quad x \in \Omega; \quad \xi = 0, \quad x \in \partial \Omega, \]
and
\[ V_t - \Delta V = 0; \quad V(0) = \xi \in L^2(\Omega). \]
Thus,

\[ tv(x, t) = W_2(x, t) - W_1(x, t) \]

where \( W_j(x, t) = V(x, \sigma_j t) \) solves

\[ W_{j, t} - \sigma_j \Delta W_j = 0. \]

We can then apply the results above on finite combinations of heat equations with different diffusivities and get

\[
\sum_k e^{-c\sqrt{\lambda_k}} |\hat{u}_{0,k}|^2 \leq C \int_0^T \int_\omega |W_1 - W_2|^2 \, dt \, dx = C \int_0^T \int_\omega |tv(x, t)|^2 \, dt \, dx
\]

\[
= C \| \int_{\sigma_1}^{\sigma_2} u(x, t; \sigma) \, d\sigma \|_{L^2(\omega \times (0, T))}^2 \leq CT^2 \| \int_{\sigma_1}^{\sigma_2} u(x, t; \sigma) \, d\sigma \|_{L^2(\omega \times (0, T))}^2
\]

Note that we do not perceive the effect of the weight \( t \) that would introduce some polynomial changes on the weights appearing in the Fourier decomposition, which is negligible with respect to the fact that weights degenerate exponentially for heat-like equations.
Accordingly, the following holds:

**Theorem**

Let \( \omega \) be an open non-empty subset of \( \Omega \) and \( T > 0 \). Assume the unknown diffusivity \( \sigma \) ranges over the set \( 0 < \sigma_1 \leq \sigma \leq \sigma_2 < \infty \). Then, the initial datum can be observed through the averages of solutions with respect to \( \sigma \) in \( \omega \times (0, T) \) on an exponentially weighted Fourier space.

**Theorem**

Let \( \omega \) be an open non-empty subset of \( \Omega \) and \( T > 0 \). Assume the unknown diffusivity \( \sigma \) of the controlled system

\[
y_t - \sigma \Delta y = f1_\omega, \quad y|_\Sigma = 0, \quad y(0, \cdot) = y^0 \in L^2(\Omega),
\]

ranges over the set \( 0 < \sigma_1 \leq \sigma \leq \sigma_2 < \infty \). Then, for all \( y^0 \in L^2(\Omega) \) there exists a control \( f \in L^2(\omega \times (0, T)) \) such that

\[
\int_{\sigma_1}^{\sigma_2} y(x, T; \sigma) d\sigma = 0.
\]
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The same result can be developed in an abstract setting\(^7\):

\[
    u_t - \sigma Au = 0, \quad t > 0; \quad u(0) = u^0,
\]

(24)

where \(A\) is the generator of a continuous semigroup in a Hilbert or Banach space.

Then

\[
    u(t, \sigma) = U(\sigma t),
\]

where \(U(\cdot, \sigma)\) solves

\[
    U_t - AU = 0, \quad t > 0; \quad U(0) = u^0(\sigma).
\]

(25)

And

\[
    \int u(t, \sigma) d\sigma = \int U(\sigma t) d\sigma = \frac{1}{t} [V(\sigma_2 t) - V(\sigma_1 t)],
\]

where \(V = V(t)\) solves the same abstract parabolic equation by with datum \(\xi = A^{-1} \xi\).

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Let us now consider the case of wave equations with an unknown velocity of propagation parameter $0 < \sigma_1 \leq \sigma \leq \sigma_2 < \infty$:

$$u_{tt} - \sigma^2 \Delta u = 0, \; u|_{\Sigma} = 0 ; u(0) = u^0, \; u_t(x, 0) = u^1. \quad (26)$$

Consider

$$v(x, t) = \int_{\sigma_1}^{\sigma_2} u(x, t; \sigma) d\sigma, \quad (27)$$

and note that

$$U_\sigma(x, t; \sigma) = u(x, t/\sigma; \sigma),$$

solves the same wave equation with $\sigma = 1$. Note however that, this time, the initial velocity of $U$ depend on the parameter $\sigma$ so that

$$U(x, 0; \sigma) = U^0(x); \; U_t(x, 0; \sigma) = \frac{1}{\sigma} u^1(x).$$
So, for the argument of the previous section to be applied directly, we rather take the initial velocity in the original equation satisfied by $u$ to be $\sigma u^1$, so that the corresponding $U$ is independent of $\sigma$ and satisfies

$$U_{tt} - \Delta U = 0; \quad U|_\Sigma = 0; \quad U(0, \cdot) = u^0, \quad U_t(x, 0) = u^1.$$  

(28)

Then,

$$u(x, t; \sigma) = U(x, \sigma t),$$

and therefore

$$v(x, t) = \int_{\sigma_1}^{\sigma_2} u(x, t; \sigma) d\sigma = \int_{\sigma_1}^{\sigma_2} U(x, \sigma t) d\sigma = \frac{1}{t} [V(x, \sigma_2 t) - V(x, \sigma_1 t)],$$

where

$$V(x, t) = \int_0^t U(x, s) ds + \xi(x)$$

$$-\Delta \xi = u^1, \quad \xi|_\Sigma = 0,$$

which is a solution of

$$V_{tt} - \Delta V = 0; \quad V|_\Sigma = 0; \quad V(0, \cdot) = \xi \in H^1_0(\Omega), \quad V_t(x, 0) = u^0(x) \in L^2(\Omega).$$
Consequently

\[ tv(x, t) = [V(x, \sigma_2 t) - V(x, \sigma_1 t)]. \]

Note that there is a regularizing effect so that \( tv \) gains one derivative with respect to the Sobolev regularity of \( u \) for each value of \( \sigma \).

Applying the previous result in collaboration with M. Lazar, the following holds:

**Theorem**

Let \( \omega \) be an open non-empty subset of \( \Omega \) and assume that \((\omega, T)\) satisfies the GCC with the fastest velocity of propagation \( \sigma_2 \).

Then, the averages of the solutions of the parameters-depending wave equation with respect to the unknown velocity of propagation \( \sigma \) is such that

\[
\| u^0 \|^2_{H^{-1}(\Omega)} + \| u^1 \|^2_{H^{-2}(\Omega)} \leq C \| t \int_{\sigma_1}^{\sigma_2} u(x, t; \sigma) d\sigma \|_{L^2(\omega \times (0, T))}^2. \quad (29)
\]
The method of proof applied to deal with additive perturbations, which consists in applying the PDE operators involved in a recursive manner, does not apply for systems of PDEs depending on a countable or continuous parameter.

The proof used to deal with additive perturbations of PDE does not apply in the context of boundary control.

The same can be said about the recent results\(^8\) about the observation of the heat equation from sets of positive measure.

The method does not apply in a straightforward manner when the coefficients in the equation are variable because of the lack of commutativity.

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In the case of wave equations can be treat general initial data without taking the velocity to be of the form $\sigma u^1$?

More general dependences on parameters, for instance for PDE involving elliptic operators of the form,

$$-\text{div}(a(x, \sigma) \nabla \cdot).$$

Can we treat the average itself, without removing the weight $1/t$ which leads, in some cases, to the loss of derivatives in the observation.

Possible connections on the theory of averaging of PDE?
• **Random dependence:** So far we considered PDE depending on deterministic unknown parameters. The same issues arise in the context of uncertain parameters governed by some probabilistic law.

• **Fourier series:** We have seen a surprising result on the possibility of extending Ingham-like inequalities to a setting where several series overlap, and all this in a robust manner. But our argument used in an essential manner the time periodicity of the wave component. The extension of these inequalities to more general hyperbolic-parabolic spectra would be of interest.
Merci de votre attention

Et merci aussi et surtout aux organisateurs!