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Controllability of Multidimensional Quasilinear Parabolic Equations, and Some Perspective on its Stochastic Version

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Part I: Controllability of Multidimensional Quasilinear Parabolic Equations

♢ Introduction

Many diffusion processes can be described by parabolic partial differential equations. If y denotes the temperature, J denotes the heat flux and f denotes the heat source, then

$$y_t + \text{ div } J = f, \quad J = -a\nabla y,$$

where a is the diffusion coefficient.

- If $a,\ f \not\sim y$: $y_t - a\triangle y = f$;
- If $f = f(y)$: $y_t - a\triangle y = f(y)$;
- If $a = a(y)$: $y_t - \text{div}(a(y)\nabla y) = f$.

Consider the following abstract system:

$$\begin{cases} y_t(t) = Ay(t) + Bu(t) & t \in (0, T), \\ y(0) = y_0, \end{cases}$$

where $u(\cdot)$ is the control variable and $y(\cdot)$ is the state variable.

- Null controllability: $\forall\, y_0 \in Y$, $\exists\, u \in L^2(0, T; U)$, s.t.
  $y(T) = 0$;

- Approximate controllability: $\forall\, y_0, y_1 \in Y$, $\forall\, \varepsilon > 0$, $\exists\, u$, s.t.
  $|y(T) - y_1|_Y < \varepsilon$.

Our goal: To develop a general method to study the above controllability properties of some quasilinear parabolic systems.

$\diamondsuit$ Some known controllability results for parabolic equations

- Linear parabolic equations

$G \subset \mathbb{R}^n$ $(n \in \mathbb{N})$: a given bounded domain.
G_0: a given nonempty open subset of G.
$Q = (0,T) \times G$, $\Sigma = (0,T) \times \partial G$.

$$\begin{cases} y_t - \sum_{j,k=1}^{n} (a^{jk} y_{x_j})_{x_k} + \sum_{j=1}^{n} (b^j y)_{x_j} + cy = \chi_{G_0} u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G. \end{cases}$$

A. V. Fursikov and O. Yu. Imanvilov (1996).

- Semilinear parabolic equations

$$\begin{cases} y_t - \displaystyle\sum_{j,k=1}^{n} (a^{jk} y_{x_j})_{x_k} + f(y) = \chi_{G_0} u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases}$$

with

$$\overline{\lim_{|s|\to\infty}} \frac{|\, f'(s) \,|}{\log^{3/2}(s)} = 0.$$

A. Doubova, E. Fernández-Cara, M. González-Burgos and E. Zuazua (2002); T. Duyckaerts, X. Zhang and E. Zuazua (2008).

- Quasilinear parabolic equations with special structures:

$$\begin{cases} y_t - \Delta(y^r) = \chi_{G_0} u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G. \end{cases}$$

$0 < r < 1$, J. I. Diaz and A. M. Ramos (1997).

$r > 1$, X. Liu and H. Gao (2007).

$$\begin{cases} y_t - \Delta y + \varepsilon \sum_{i,j=1}^{n} g_{ij}(y, \nabla y) y_{x_i x_j} = \chi_{G_0} u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G. \end{cases}$$

E. Fernández-Cara and S. Guerrero (2006).

- Quasilinear parabolic equations in one dimension:

$$\begin{cases} y_t - (a(y))_{xx} = \chi_{G_0} u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases}$$

where $a(\cdot) \in C^3(\mathbb{R})$, $\sup_{s\in\mathbb{R}}\{|a'(s)|, |a''(s)|, |a'''(s)|\} < \infty$ and $\inf_{s\in\mathbb{R}} a'(s) > 0$.

M. Beceanu (2003).

Remark. The method used in the one dimensional case is not applicable to the multidimensional case directly. Indeed, consider the following linearized system:

$$y_t - \text{div}(a(z)\nabla y) = \chi_{G_0} u \quad \text{in } Q.$$

- Carleman estimates for the linear parabolic equation require that $z \in W^{1,\infty}(Q)$.
- When $n = 1$, $L^\infty(0, T; H_0^1(G)) \hookrightarrow L^\infty(Q)$ + $u, u_t \in L^2(Q)$ $\Longrightarrow$ If $|z|_{W^{1,\infty}(Q)} \leq C^*$, then $|y|_{W^{1,\infty}(Q)} \leq C^*$, for sufficiently small y_0.
- When $n > 1$, the above embedding relation fails.

$\diamondsuit$ Controllability of quasilinear parabolic equations

Consider the following quasilinear parabolic equation:

$$\begin{cases} y_t - \sum_{j,k=1}^{n} (a^{jk}(y) y_{x_j})_{x_k} + f(y) = \chi_{G_0} u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \tag{1}$$

where $a^{jk}(\cdot) \in C^3(\mathbb{R})$, $a^{jk} = a^{kj}$, $f \in C^2(\mathbb{R})$, $f(0) = 0$ and for some $\rho > 0$,

$$\sum_{j,k=1}^{n} a^{jk}(s) \xi_j \overline{\xi_k} \geq \rho |\xi|^2, \ \forall (s, \xi) \equiv (s, \xi_1, \cdots, \xi_n) \in \mathbb{R} \times \mathbb{C}^n.$$

Theorem (X. Liu and X. Zhang, 2012) There is a positive constant δ_1, such that for any given initial value $y_0 \in C^{2+\frac{1}{2}}(\overline{\Omega})$ satisfying $|y_0|_{C^{2+\frac{1}{2}}(\overline{G})} \le \delta_1$ and the first order compatibility condition, one can find a control $u \in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$ with $\operatorname{supp} u \subseteq G_0 \times [0,T]$ so that the corresponding solution y of the system (1) satisfies $y(T) = 0$ in G. Moreover,

$$|u|_{C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})} \le Ce^{e^{CA_1}}|y_0|_{L^2(G)},$$

where

$$A_1 = 1 + \sum_{j,k=1}^{n} |a^{jk}|^2_{C^1([-1,1])} + |f'|^2_{C([-1,1])}.$$

Sketch of a proof. Use the fixed point technique in the frame of classical solutions. Define

$$K = \{z \in C^{1+\frac{1}{2},1+\frac{1}{4}}(\overline{Q});\ |z|_{C^{1+\frac{1}{2},1+\frac{1}{4}}(\overline{Q})} \leq 1,\ z(0) = y_0\}.$$

- For $z \in K$, the linearized system of (1) is null controllable, by establishing an explicit Carleman estimate in terms of the C^1 coefficients on the principal operator.

- If $u \in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$, by the Schauder estimate of linear parabolic equations, when $|y_0|_{C^{2+\frac{1}{2}}(\overline{G})}$ is sufficiently small, $y \in K$.

The key: To improve the regularity of the control and find a control in the space $C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$ for linear parabolic equations.

To this aim, consider the following linear system:

$$\begin{cases} y_t - \sum_{j,k=1}^{n} (b^{jk} y_{x_j})_{x_k} + dy = \zeta u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \tag{2}$$

where $b^{jk}(\cdot) \in C^1(\overline{Q})$, $b^{jk} = b^{kj}$, $d \in L^\infty(Q)$, $\zeta \in C_0^\infty(G_0)$, $\zeta \equiv 1$ in G_1, and $\sum_{j,k=1}^{n} b^{jk}(x,t)\xi_j\overline{\xi_k} \geq \rho|\xi|^2$, $\forall (x,t,\xi) \in \overline{Q} \times \mathbb{C}^n$.

Proposition. For any $y_0 \in L^2(\Omega)$, there exists a control $u \in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$, such that the corresponding solution y of (2) satisfies $y(T) = 0$ in G.

Idea: To adopt an iteration technique (V. Barbu, 2002).

- Construct suitable optimal control problems $\Longrightarrow$ The optimal controls are a sequence of "approximate" null-controls for the linear system. Moreover, these controls can be represented by the solutions of adjoint equations and certain weight functions.

- By an iteration technique and the L^p estimates for linear parabolic equations, the solutions of adjoint equations are uniformly bounded in the space $C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$, if multiplied by a suitable weight function.

  $\Longrightarrow$ The controls are uniformly bounded in $C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$.

$\diamondsuit$ Other control problems for quasilinear parabolic systems

• Controllability of coupled quasilinear parabolic systems (H. Li and X. Liu, 2014)

$$
\begin{cases}
y_{1,t} - \displaystyle\sum_{j,k=1}^{n} \Big(a_1^{jk}(y) y_{1,x_j} \Big)_{x_k} + f_1(y_1, \cdots, y_m) = \chi_{G_0} u & \text{in } Q, \\
y_{2,t} - \displaystyle\sum_{j,k=1}^{n} \Big(a_2^{jk}(y) y_{2,x_j} \Big)_{x_k} + f_2(y_1, \cdots, y_m) = 0 & \text{in } Q, \\
y_{3,t} - \displaystyle\sum_{j,k=1}^{n} \Big(a_3^{jk}(y) y_{3,x_j} \Big)_{x_k} + f_3(y_2, \cdots, y_m) = 0 & \text{in } Q, \\
\cdots\cdots & \\
y_{m,t} - \displaystyle\sum_{j,k=1}^{n} \big(a_m^{jk}(y) y_{m,x_j} \big)_{x_k} + f_m(y_{m-1}, y_m) = 0 & \text{in } Q, \\
y_1 = \cdots = y_m = 0 & \text{on } \Sigma, \\
y_1(0) = y_1^0, \cdots, y_m(0) = y_m^0 & \text{in } G.
\end{cases}
$$

• Controllability of quasilinear Ginzburg-Landau equations (X. Fu, X. Liu and X. Zhang, 2014)

$$
\begin{cases}
(a_1 - ia_2)y_t - \displaystyle\sum_{j,k=1}^{n} (a^{jk}(|y|^2)y_{x_j})_{x_k} & \\
\qquad\qquad + f(|y|^2)y = \chi_{G_0} u & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0 & \text{in } G,
\end{cases}
$$

where $a_1 > 0,\ a_2 \neq 0$ and $i = \sqrt{-1}$. Both the control variable u and the state variable y are complex valued.

• Insensitizing controls for quasilinear parabolic equations (X. Liu, 2012)

Part II: Some New Phenomenon and difficulties for the Stochastic Controllability

◇ Controllability for stochastic ODEs

Consider the following controlled (ODE) system:

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, & t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $T > 0$. System (3) is said to be controllable on $(0,T)$ if for any $y_0, y_1 \in \mathbb{R}^n$, there exists a $u \in L^1(0,T;\mathbb{R}^m)$ such that $y(T) = y_1$.

Put

$$G_T = \int_0^T e^{At} B B^* e^{A^* t} dt.$$

Theorem: If the system (3) is controllable on $(0, T)$, then $\det G_T \neq 0$. Moreover, for any $y_0, y_1 \in \mathbb{R}^n$, the control

$$u^*(t) = -B^* e^{A^*(T-t)} G_T^{-1} (e^{AT} y_0 - y_1)$$

transfers y_0 to y_1 at time T.

Clearly, if (3) is controllable on $(0, T)$ (by means of L^1-(in time) controls), then the same controllability can be achieved by using analytic-(in time) controls. We shall see a completely different phenomenon in the simplest stochastic situation.

• The stochastic setting

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$: a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined.

H: a Banach space, and write $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$.

$L^2_{\mathbb{F}}(0, T; H)$: the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm;

Similaryly, $L^\infty_{\mathbb{F}}(0, T; H)$, $L^2_{\mathbb{F}}(\Omega; C([0, T]; H))$, etc.

Consider a one-dimensional controlled stochastic differential equation:

$$dx(t) = [bx(t) + u(t)]dt + \sigma dB(t), \tag{4}$$

with b and σ being given constants. We say that the system (4) is exactly controllable if for any $x_0 \in \mathbb{R}$ and $x_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$, there exists a control $u(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0, T; \mathbb{R}))$ such that the corresponding solution $x(\cdot)$ satisfies $x(0) = x_0$ and $x(T) = x_T$.

It is shown by Q. Lü, J. Yong and X. Zhang (2012) that the system (4) is exactly controllable at any time $T > 0$ (by means of $L^2_{\mathbb{F}}(\Omega; L^1(0, T; \mathbb{R}))$-controls).

On the other hand, surprisingly, in virtue of a result by S. Peng (1994), the system (4) is NOT exactly controllable if one restricts to use admissible controls $u(\cdot)$ in $L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}))$!

It is shown also by Q. Lü, J. Yong and X. Zhang (2012) that the system (4) is NOT exactly controllable, either provided that one uses admissible controls $u(\cdot)$ in $L^2_{\mathbb{F}}(\Omega; L^q(0,T;\mathbb{R}))$ for any $q \in (1,\infty]$. This leads to a corrected formulation for the exact controllability of stochastic differential equations, as presented below.

Consider a linear stochastic differential equation:

$$\begin{cases} dy = \big(Ay + Bu\big)dt + \big(Cy + Du\big)dB(t), \quad t \geq 0, \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases} \tag{5}$$

where $A, C \in \mathbb{R}^{n\times n}$ and $B, D \in \mathbb{R}^{n\times m}$. Unlike the deterministic case, there exists no universally accepted notion for stochastic controllability so far.

Motivated by the above observation, we introduce the following definition: System (5) is said to be exactly controllable if for any $y_0 \in \mathbb{R}^n$ and $y_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, there exists a control $u(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0, T; \mathbb{R}^m))$ such that $Du(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$ and the corresponding solution $y(\cdot)$ of (5) satisfies $y(T) = y_T$.

Though the above definition seems to be a reasonable notion for exact controllability of stochastic differential equations, a complete study on this problem is still under consideration and it does not seem to be easy.

When $n > 1$, the controllability for the linear system (5) is in general unclear.

Compared to the deterministic case, the controllability/ observability for stochastic differential equations is in its "enfant" stage.

♢ Controllability for stochastic parabolic equations

• Null controllability of stochastic parabolic equations with two controls:

$$\begin{cases} dy - \displaystyle\sum_{i,j=1}^{n}(a^{ij}y_i)_j dt \\ \qquad = [\langle\, \alpha, \nabla y\, \rangle + \beta y + \chi_{G_0}\gamma]dt + (qy + \Gamma)\, dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{6}$$

The null controllability of (6): For any given $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, one can find a control $(\gamma, \Gamma) \in L^2_{\mathbb{F}}(0, T; L^2(G_0)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$ such that the solution y of (6) satisfying $y(T) = 0$ in G, P-a.s.

By means of the classical duality argument, the null controllability of (6) may be reduced to the observability estimates for the following backward stochastic parabolic equations:

$$\begin{cases} dz + \sum_{i,j=1}^{n} (a^{ij} z_i)_j dt \\ \qquad = [\langle a, \nabla z \rangle + bz + cZ]dt + ZdB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = z_T & \text{in } G, \end{cases} \tag{7}$$

i.e., to find a constant $C > 0$ such that all solutions of (7) satisfy

$$|z(0)|_{L^2(\Omega,\mathcal{F}_0,P;L^2(G))} \leq C \left(|z|_{L^2_{\mathcal{F}}(0,T;L^2(G_0))} + |Z|_{L^2_{\mathcal{F}}(0,T;L^2(G))} \right), \quad \forall\, z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G)). \tag{8}$$

S. Tang and X. Zhang (2004, 2009) proved (8), by the following identity for a stochastic parabolic-like operator:

Theorem Let $b^{ij} = b^{ji} \in C^{1,2}$ $(i, j = 1, 2, \cdots, m)$, $\ell \in C^{1,3}$, u be an $C^2(\mathbb{R}^m)$-valued semimartingale. Set $\theta = e^{\ell}$ and $v = \theta u$. Then, for a suitable function $\mathcal{M}$,

$$
\begin{aligned}
&2\int_0^T \theta\Big[-\sum_{i,j=1}^m (b^{ij}v_{x_i})_{x_j} + Av\Big]\Big[du - \sum_{i,j=1}^m (b^{ij}u_{x_i})_{x_j}dt\Big] \\
&\quad + \int_0^T \sum_{j=1}^m \Big[\cdots\Big]_{x_j} dt + 2\int_0^T \sum_{i,j=1}^m (b^{ij}v_{x_i}dv)_{x_j} \\
&= \int_0^T \sum_{i,j=1}^m \Big\{\cdots\Big\} v_{x_i}v_{x_j}dt + \int_0^T (\cdots)v^2 dt + \cdots \\
&\quad + \Big(\cdots\Big)\Big|_0^T - \int_0^T \theta^2 \sum_{i,j=1}^m b^{ij}du_{x_i}du_{x_j} - \int_0^T \theta^2 \mathcal{M}(du)^2.
\end{aligned}
$$

• Controllability of stochastic parabolic equations with one control:

$$
\begin{cases}
dy - \sum_{i,j=1}^{n} (a^{ij}(x) y_{x_i})_{x_j} dt = a(t) y dB(t) + \chi_E \chi_{G_0} u dt & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0 & \text{in } G,
\end{cases}
\tag{9}
$$

where $a(t) \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R})$, E is a measurable subset in $(0, T)$ with $m(E) > 0$, $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, the control u belongs to $L^{\infty}_{\mathcal{F}}(0, T; L^2(\Omega; L^2(G)))$.

Q. Lü (2011) proved the following two results:

(i) System (9) is null controllable at time T;

(ii) System (9) is approximately controllable at time T if and only if $m((s,T)\bigcap E)>0$ for any $s\in[0,T)$.

Surprisingly, one needs more assumptions in (ii) for the approximate controllability (of (9)) than that in (i) for the null controllability. Indeed, it is well known that in the deterministic setting, the null controllability is usually stronger than the approximate controllability. But this does not remain to be true in the stochastic case.

This indicates that there exists some essential difference between the controllability theory between deterministic heat equations and stochastic heat equations.

• Controllability of some coupled stochastic parabolic systems by one control:

$$
\begin{cases}
dy - \Delta y dt = [a(t)y + b(t)z + \chi_{G_0} u]\, dt \\
\qquad\qquad\qquad + [c(t)y + f(t)z]\, dB(t) & \text{in } Q, \\
dz - \Delta z dt = [d(t)y + e(t)z]\, dt \\
\qquad\qquad\qquad + [g(t)z + h(t)y]\, dB(t) & \text{in } Q, \\
y = 0, \ z = 0 & \text{on } \Sigma, \\
y(0) = y_0, \ z(0) = z_0 & \text{in } G,
\end{cases}
\tag{10}
$$

where u is the control variable, (y, z) is the state variable, $(y_0, z_0) \in \big(L^2(\Omega, \mathcal{F}_0, \mathcal{P}; L^2(G))\big)^2$ is any given initial value. Moreover, $a,\ b,\ c,\ d,\ e,\ f,\ g,\ h \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$ are given coefficients.

Suppose that the coefficients d and h satisfy:

(H_1) There exist a positive constant d_0 and an interval $I \subset [0, T]$ such that $d(t) \geq d_0$ or $d(t) \leq -d_0$ on I;

(H_2) There exists an interval $\widetilde{I} \subseteq I$ such that $h(t) = 0$ on $\widetilde{I}$.

Theorem (X. Liu, 2014): Under the conditions $(\mathbf{H_1})$ and $(\mathbf{H_2})$, the system (10) is null controllable.

It deserve to emphasize that neither $(\mathbf{H_1})$ nor $(\mathbf{H_2})$ in the above Theorem could be dropped, as shown by X. Liu (2014).

Especially, by virtue of some counterexamples by X. Liu (2014), it is found that the controllability of the coupled system (10) is not robust with respect to the coupling coefficient in the diffusion terms.

Indeed, when this coefficient equals to zero on the whole time interval $[0, T]$, the system is null controllable. However, if it is a nonzero bounded function, no matter how small this function is, the corresponding system is uncontrollable any more!

This indicates that the Carleman-type estimates cannot be used to study the controllability of the coupled stochastic parabolic system (10). Note however that, one can use the Carleman estimate to prove the controllability of the deterministic version of (10).

Part III: Open problems

- The controllability for the following more general quasilinear parabolic equation:

$$y_t - \sum_{j,k=1}^{n} (a^{jk}(y, \nabla y) y_{x_j})_{x_k} = \chi_{G_0} u.$$

- The global controllability for quasilinear parabolic equations.

- The case when the nonlinearity $a^{jk}(\cdot)$ has less regularity.

- Stabilization problems.
- Numerical approaches.
- The degenerate cases.
- The fully nonlinear problems (Solutions in the sense of viscosity?)
- The stochastic version!

Thank you !