Approximation of periodic solutions for a dissipative hyperbolic equation

Sorin Micu

University of Craiova (Romania)

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Introduction

Andrew M. Stuart, Numerical analysis of dynamical systems, Acta Numerica (1994), pp. 467-572.

"To understand the behavior of **the approximation of initial value problems over long-time intervals**, standard error estimates for individual trajectories are of no direct use since the error constant typically grows like the exponential of the time interval under consideration.

A deeper knowledge of the behavior of the continuous solutions and, in particular, an understanding of how these solutions behave over long-time intervals is very useful. In particular, **the study of a variety of sets invariant under the evolution generated by the equation** (equilibrium points, periodic solutions, quasi-periodic solutions and strange attractors) is crucial.

Such knowledge, combined with the standard error estimate or a truncation error bound, can provide very powerful results about the long-time behavior of numerical methods."

Exponentially stable hyperbolic equations

Let H and U be two Hilbert spaces.

- $A : \mathcal{D}(A) \to H$ is a self-adjoint, strictly positive operator with compact resolvent.
- $B \in \mathcal{L}(U, H)$ is an input operator.

$$\ddot{u}(t) + Au(t) + BB^*\dot{u}(t) = 0 \qquad (t > 0). \tag{1}$$

The energy of (1) is defined by

$$E(t) = \frac{1}{2}\|(u(t), \dot{u}(t))\|^2_{\mathcal{D}(A^{\frac{1}{2}})\times H}. \tag{2}$$

It follows that

$$E(0) - E(t) = \int_0^t \|B^*\dot{u}(s)\|_U^2 ds. \tag{3}$$

Exponentially stable hyperbolic equations

We suppose that there exist two positive constants M and ω such that, for every solution $(u, \dot{u})$ of (1) with initial data $(u(0), \dot{u}(0)) = (u^0, u^1) \in \mathcal{D}(A^{\frac{1}{2}}) \times H$, the following holds

$$E(t) \leqslant M^2 e^{-2\omega t} E(0). \tag{4}$$

- The decay of the energy is exponential (uniform).
- Each solution $(u(t), \dot{u}(t))$ of (1) tends exponentially to the unique equilibrium point $(0, 0)$ as t goes to infinity.

Periodic solutions

Let $f \in \mathcal{C}([0,\infty);H)$ be a time periodic function of period T

$$f(t+T,\cdot\,) = f(t,\cdot\,) \qquad (t \geqslant 0) \tag{5}$$

We consider the non-homogeneous system

$$\ddot{u}(t) + Au(t) + BB^*\dot{u}(t) = f(t) \qquad (t > 0). \tag{6}$$

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$$\ddot{u}(t) + Au(t) + BB^*\dot{u}(t) = f(t) \qquad (t > 0). \tag{6}$$

Main questions:

- **Does there exist a $T-$periodic solution $(\widehat{u}, \dot{\widehat{u}})$ of (6)?**
- **How can it be approximated?**

Remarks

- By taking into account the stability of (1) it follows that, if it exists, $(\widehat{u}, \dot{\widehat{u}})$ is unique and it is a global attractor for (6): For any other solution $(u, \dot{u})$ of (6) we have that:

$$\lim_{t \to \infty} \|(\widehat{u}(t), \dot{\widehat{u}}(t)) - (u(t), \dot{u}(t))\|_{\mathcal{D}(A^{\frac{1}{2}}) \times H} = 0.$$

- The existence of a periodic solution is equivalent to the boundedness of all trajectories (and, consequently, the lack of the resonance phenomenon).

Vectorial form

With the new variable $v = \dot{u}$, (6) is equivalently written as

$$\dot{U}(t) = \mathbb{A}U(t) + F(t) \qquad (t > 0) \tag{7}$$

where

$$U(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \ \mathbb{A} = \begin{bmatrix} 0 & I \\ -A & -BB^* \end{bmatrix}, \ F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

- For each $\alpha \geqslant 0$, define the Hilbert space $X_\alpha = H_{\alpha+\frac{1}{2}} \times H_\alpha$.
- The operator $(X_{\frac{1}{2}}, \mathbb{A})$ is the infinitezimal generator of a contraction semigroup $\mathbb{S}$ on $X := X_0$.
- The semigroup $\mathbb{S}$ is exponentially stable in X and $X_{\frac{1}{2}}$. If, in addition, $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$, then $\mathbb{S}$ is an exponentially stable semigroup in X_1, too.

Existence of periodic solutions

Theorem 1

There exists a unique $(\widehat{u}^0, \widehat{u}^1) \in X$ *such that the corresponding mild solution of* (7) *in* X *with initial data* $(\widehat{u}^0, \widehat{u}^1)$ *is* $T-$*periodic. If, in addition,* $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ *and* $f \in W^{1,1}_{loc}([0,\infty); H_{\frac{1}{2}})$ *or* $f \in L^1_{loc}([0,\infty); H_1)$*, then there exists a unique* $(\widehat{u}^0, \widehat{u}^1) \in X_1$ *such that the corresponding classical solution of* (7) *in* X_1 *with initial data* $(\widehat{u}^0, \widehat{u}^1)$ *is* $T-$*periodic.*

Proof of Theorem 1

We introduce the map $\Lambda : X \to X$ defined by

$$\Lambda U^0 = U(T) = \mathbb{S}(T)U^0 + \int_0^T \mathbb{S}(T-s)F(s)\,ds. \qquad (8)$$

We have that, for any U^0 and U^1 in X,

$$\|\Lambda^n U^0 - \Lambda^n U^1\|_X = \|S(nT)(U^0 - U^1)\|_X \leqslant Me^{-n\omega T}\|U^0 - U^1\|_X.$$

For n sufficiently large, Λ^n is a contraction on X and therefore there exists a unique $\widehat{U}^0 \in X$ such that $\Lambda^n \widehat{U}^0 = \widehat{U}^0$. Consequently, $\Lambda^n(\Lambda\widehat{U}^0) = \Lambda(\Lambda^n\widehat{U}^0) = \Lambda\widehat{U}^0$ and, by the uniqueness of the fixed point of Λ^n, we obtain

$$\Lambda\widehat{U}^0 = \widehat{U}^0.$$

In the sequel we denote $q = e^{-\omega T}$.

References

- Haraux, Alain: Semi-linear hyperbolic problems in bounded domains, Mathematical Reports 3, 1987.
- Bristeau, Marie-Odile, Glowinski, Roland and Périaux, Jacques: Controllability methods for the computation of time-periodic solutions, Journal of Computational Physics, 147 (1998), 265-292.
- Bardos, Claude and Rauch, Jeffrey: Variational algorithms for the Helmholtz equation using time evolution and artificial boundaries, Asymptotic Analysis, 9 (1994), 101-117.
- Zuazua, Enrique: On the numerical approximation of the Helmholtz equations, Matemática Contemporânea, 32 (2007), 253-286.

Motivation: Helmholtz equation

$$\ddot{u}(t,x) - \Delta u(t,x) = e^{i\,\sigma\,t} g(x) \quad x \in \Omega,\ t > 0 \tag{9}$$

has a periodic solution $u = e^{i\,\sigma\,t} w$ if and only if w verifies the Helmholtz's equation

$$(\sigma^2 + \Delta) w(x) = -g(x) \quad x \in \Omega. \tag{10}$$

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- If we are interested in the Helmholtz's equation in an exterior domain $\Omega \subset \mathbb{R}^n$, for numerical reasons, the unbounded domain Ω has to be limited by introducing an artificial boundary $\partial\Omega$ with a Sommerfeld radiation condition (1912)

$$\partial_n u(x) - iku(x) = 0 \quad x \in \partial\Omega. \tag{11}$$

- Since solving directly (10) leads to large indefinite linear systems, alternative methods for finding the solutions of the Helmholtz's equation have been proposed: one of them consists in obtaining the periodic solution of (9).

Motivation: Periodic solutions by minimization

We define the functional

$$J(U^0) = \frac{1}{2}\|U(T) - U^0\|_X^2 \qquad (U^0 \in X)$$

where U is the solution of (7).

- Finding the periodic solution $\widehat{U}$ of (7) is equivalent to find the minimum $\widehat{U}^0$ of J: $\widehat{U}(0) = \widehat{U}^0$
- The coerciveness of J requires the exponential decay of the energy. The coerciveness of J and the contractive properties of the operator Λ are both due to the exponential decay.
- A slightly more complicated functional (involving an additional time integration) introduced by Bardos and Rauch only needs the strong stabilization property for the coerciveness.
- After discretization, the exponential decay is not uniform with respect to the mesh size and the coerciveness of J and the contractive properties of Λ become very weak.

Arnold Johannes Wilhelm Sommerfeld (1868-1951)

"We do not really deal with mathematical physics, but with physical mathematics; not with the mathematical formulation of physical facts, but with the physical motivation of mathematical methods. The oft-mentioned "prestabilized harmony" between what is mathematically interesting and is physically important is met at each step and lends an esthetic-I should like to say metaphysical-attraction to our subject." (Sommerfeld 1945)

Discrete setting

Assume that there exists a family $(V_h)_{h>0}$ of finite dimensional subspaces of $H_{\frac{1}{2}}$ and that there exist $\theta > 0$, $h^* > 0$, $C_0 > 0$ such that, for every $h \in (0, h^*)$,

$$\|\pi_h \varphi - \varphi\|_{\frac{1}{2}} \leq C_0\, h^\theta \|\varphi\|_1 \qquad (\varphi \in H_1), \tag{12}$$

$$\|\pi_h \varphi - \varphi\| \leq C_0\, h^\theta \|\varphi\|_{\frac{1}{2}} \qquad (\varphi \in H_{\frac{1}{2}}), \tag{13}$$

where π_h is the orthogonal projector from $H_{\frac{1}{2}}$ onto V_h.

Assumptions (12)-(13) are, in particular, satisfied when finite elements are used for the approximation of Sobolev spaces. The inner product in V_h is the restriction of the inner product on H and it is still denoted by $\langle \cdot, \cdot \rangle$. $N(h)$ denotes the dimension of V_h.

Discrete setting

We define the linear operator $A_h \in \mathcal{L}(V_h)$ by

$$\langle A_h \varphi_h, \psi_h \rangle = \langle A^{\frac{1}{2}} \varphi_h, A^{\frac{1}{2}} \psi_h \rangle \qquad (\varphi_h,\ \psi_h \in V_h). \tag{14}$$

Let $U_h = B^* V_h \subset U$ and define the operator $B_h \in \mathcal{L}(U, H)$ by

$$B_h u = \widetilde{\pi}_h B u \qquad (u \in U), \tag{15}$$

where $\widetilde{\pi}_h$ is the orthogonal projection from H onto V_h.
The adjoint $B_h^* \in \mathcal{L}(H, U)$ of B_h is

$$B_h^* \varphi = B^* \widetilde{\pi}_h \varphi \qquad (\varphi \in H). \tag{16}$$

Since $U_h = B^* V_h$, from (16) it follows that $\operatorname{Ran} B_h^* = U_h$ and that

$$\langle B_h^* \varphi_h, B_h^* \psi_h \rangle_U = \langle B^* \varphi_h, B^* \psi_h \rangle_U \qquad (\varphi_h,\ \psi_h \in V_h). \tag{17}$$

The above assumptions imply that, for every $h^* > 0$, the family $\left(\|B_h\|_{\mathcal{L}(U_h, H)} \right)_{h \in (0, h^*)}$ is bounded.

Semi-discrete equation

We consider the following semi-discrete scheme for (6)

$$\begin{cases} \ddot{u}_h(t) + A_h u_h(t) + B_h B_h^* \dot{u}_h(t) = f_h(t), & (t > 0) \\ u_h(0) = u_{0h},\ \dot{u}_h(0) = u_{1h}, \end{cases} \tag{18}$$

where $f_h \in \mathcal{C}([0,\infty); V_h)$ and $(u_{0h}, u_{1h}) \in V_h^2$.
We consider the product space $X_h = V_h \times V_h$ equipped with the inner product

$$\left\langle \begin{bmatrix} \varphi_{1h} \\ \psi_{1h} \end{bmatrix}, \begin{bmatrix} \varphi_{2h} \\ \psi_{2h} \end{bmatrix} \right\rangle_{X_h} = \langle A_h^{\frac{1}{2}} \varphi_{1h}, A_h^{\frac{1}{2}} \varphi_{2h} \rangle + \langle \psi_{1h}, \psi_{2h} \rangle,$$

and we denote by $\| \cdot \|_{X_h}$ the corresponding norm in X_h.

Semi-discrete equation

Using the new variable $v_h = \dot{u}_h$, system (18) can be rewritten in X_h as

$$\dot{U}_h(t) = \mathbb{A}_h U_h(t) + F_h(t), \quad (t > 0) \tag{19}$$

where $\mathbb{A}_h = \mathbb{A}_h^1 - \mathbb{B}_h$ and

$$U_h(t) = \begin{bmatrix} u_h(t) \\ v_h(t) \end{bmatrix}, \ \mathbb{A}_h^1 = \begin{bmatrix} 0 & I \\ -A_h & 0 \end{bmatrix}, \ \mathbb{B}_h = \begin{bmatrix} 0 \\ B_h B_h^* \end{bmatrix}, \ F_h(t) = \begin{bmatrix} 0 \\ f_h(t) \end{bmatrix}.$$

Note that $\mathbb{A}_h^1$ is a skew-adjoint operator on X_h and $\mathbb{A}_h \in \mathcal{L}(X_h)$ is the infinitesimal generator of a uniformly continuous semigroup of linear bounded operators $\mathbb{S}_h(t) = e^{t\mathbb{A}_h}$ on X_h. Moreover, since $\mathbb{A}_h$ is m-dissipative, the semigroup $\mathbb{S}_h$ is of contractions in X_h.

Discrete energy

The discrete energy corresponding to (18) is defined by

$$E_h(t) = \frac{1}{2}\left(\|A_h^{\frac{1}{2}} u_h\|^2 + \|\dot{u}_h\|^2\right). \tag{20}$$

If $f_h = 0$, we deduce that

$$\frac{dE_h}{dt}(t) = -\|B^* \dot{u}_h(t)\|_U^2 \qquad (t \geqslant 0).$$

In the sequel we shall suppose that the following hypothesis holds

$$\lim_{t\to\infty} E_h(t) = 0. \tag{21}$$

An error estimate

Hypothesis

- $f_{|_{[0,T]}} \in W^{1,1}(0,T;H_{\frac{1}{2}})$ and f is a $T-$periodic function.
- $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$
- $\lim_{t\to\infty} E_h(t) = 0$

Proposition 1

Let $(w_0, w_1) \in X_1$, $(w_h^0, w_h^1) = (\pi_h w^0, \pi_h w^1)$ *and* $f_h(t) = \pi_h f(t)$. *If* w *and* w_h *are the solutions of* (6) *and* (18), *respectively, then there exist* $K_0,\ K_1 > 0$ *such that, for each* $t \in (0,T)$,

$$\|(w,\dot{w})(t) - (w_h,\dot{w}_h)(t)\|_X \leqslant (K_0 + K_1\,T)h^\theta \left(\|(w_0,w_1)\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right). \quad (22)$$

Decay of the discrete energy

Proposition 2

Let $h > 0$. Suppose that $f_h \equiv 0$ and (21) holds. There exist two positive constants M independent of h and $\omega = \omega(h)$ such that

$$E_h(t) \leqslant M^2 E_h(0) e^{-2\omega(h)t} \qquad (t \geqslant 0 \qquad (u_{0h}, u_{1h}) \in V_h^2). \quad (23)$$

It is known that, in general, the decay rate is not uniform with respect to h:

$$\omega(h) \to 0 \text{ as } h \to 0.$$

In the sequel we denote $q_h = e^{-\omega(h)T}$.

Existence of the discrete periodic solutions

Theorem 2

There exists a unique $\widehat{U}_h^0 \in V_h^2$ *such that the corresponding solution* $\widehat{U}_h \in \mathcal{C}^1\left([0,\infty); V_h^2\right)$ *of* (19) *with initial data* $\widehat{U}_h^0$ *is* $T-$*periodic.*

Proof: We define the operator $\Lambda_h : V_h^2 \to V_h^2$ by

$$\Lambda_h U_h^0 = \mathbb{S}_h(T) U_h^0 + \int_0^T \mathbb{S}_h(T-s) F_h(s)\, ds, \tag{24}$$

where $\{\mathbb{S}_h(t)\}_{t\geqslant 0}$ is the semigroup generated by $\mathbb{A}_h$ in X_h. We have

$$\|\Lambda_h U_h^0 - \Lambda_h U_h^1\|_{X_h} \leqslant M\, e^{-\omega(h)T} \|U_h^0 - U_h^1\|_{X_h}.$$

Now, the argument follows as in Theorem 1.

Approximation of the periodic solutions. Algorithm

We analyze the following algorithm for the approximation of the periodic solutions:

- Choose $h > 0$ small and $U_h^0 \in V_h^2$;
- Choose an integer $N > 0$ and a precision $\epsilon > 0$;
- Compute $\Lambda_h^n U_h^0$ until

$$\|\Lambda_h^n U_h^0 - \Lambda_h^{n-1} U_h^0\| < \epsilon$$

or

$$n > N;$$

- Approximate $\widehat{U}^0$ by $\Lambda_h^n U_h^0$.

The error estimates will show how the numbers N and ϵ should be chosen.

The above algorithm combines two approximation processes:

- One for computing the solution of the wave equation at time T, given by $\Lambda_h(u_h^0, u_h^1)$. It depends on the Galerkin scheme and the discrete operators A_h and B_h.
- Another one for determining the fixed point of Λ_h. It is governed by the contractive properties of Λ_h.

Most often, improving one of these approximations leads to the deterioration of the other one.

The non uniform (with respect to the mesh-size h) decay property of the solutions of the discrete equation leads to a degeneracy of the contractive properties of the operator Λ_h as h tends to zero.

The Screenplay

Three main problems will be investigated:

(P1) Does the family $\left(\Lambda_h^n U_h^0\right)_{h>0,\, n\geqslant 0}$ converge to the fixed point $\widehat{U}^0$ of Λ when h tends to zero and n tends to infinity and, if it does, at which rate?

(P2) Does the family $\left(\widehat{U}_h^0\right)_{h>0}$ of fixed points of the discrete operators Λ_h converges to the fixed point $\widehat{U}^0$ of Λ when h goes to zero and, if it does, at which rate?

(P3) How do the two convergence properties mentioned above change if a numerical vanishing viscosity is introduced in the equation or if special monochromatic nonhomogeneous terms f are considered?

Approximation of the periodic solutions. Convergence

Theorem 3

Let $\widehat{U}^0$ be the unique fixed point of Λ from Theorem 1 and $\widehat{U}_h^0$ be the fixed point of the discrete operator Λ_h Theorem 2. Then there exists a constant $C > 0$ such that, for each $U^0 \in X_1$, $n \geqslant 1$ and $h < h^*$, the following estimates hold

$$\|\widehat{U}^0 - \Lambda_h^n(\Pi_h U^0)\|_X \leqslant C\left(n\,h^\theta + \frac{q^n}{1-q}\right)\left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right)$$

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant C\left(n\,h^\theta + \frac{q^n}{1-q} + \frac{q_h^n}{1-q_h}\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}.$$

We recall that $q = e^{-\omega T} < 1$ and $q_h = e^{-\omega(h)T}$ (it may tend to 1 as h goes to zero).

Remarks

- From Theorem 3 we deduce:
  - the convergence of the sequence of iterations $\left(\Lambda_h^n(\Pi_h U^0)\right)_{n\geqslant 0}$
  - the convergence of the fixed points of Λ_h to the fixed point of Λ may fail.
- The convergence rate is slightly worse than h^θ:

$$n = \left[\frac{\theta}{|\ln(q)|}\ln\left(\frac{1}{h}\right)\right] + 1 \Rightarrow n\, h^\theta + \frac{q^n}{1-q} \;\sim\; h^\theta \ln\left(\frac{1}{h}\right).$$

Proof of Theorem 3

Let $U^0 \in X_1 = H_{\frac{3}{2}} \times H_1$, $U_h^0 = \Pi_h U^0$ and $n \geqslant 1$.

$$\|\widehat{U}^0 - \Lambda_h^n(U_h^0)\|_X$$

$$\leqslant \|\widehat{U}^0 - \Lambda^n U^0\|_X + \|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X + \|\Pi_h \Lambda^n U^0 - \Lambda_h^n \Pi_h U^0\|_X.$$

$$\|\widehat{U}^0 - \Lambda^n U^0\|_X \leqslant \frac{Mq^n}{1-q}\left[\|U^0\|_X + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right]$$

(fixed point iterations)

$$\|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X \leqslant Ch^\theta \|\Lambda^n U^0\|_X \leqslant Ch^\theta \left[\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right]$$

(finite elements estimate)

$$\|\Pi_h \Lambda^n U^0 - \Lambda_h^n \Pi_h U^0\|_X \leqslant MCh^\theta\, n\, \left[\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right].$$

(error estimate for solutions+contraction property of Λ_h)

Proof of Theorem 3

Let $U^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in X_1$, $U_h^0 = \Pi_h U^0$ and $n \geqslant 1$. We deduce that

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant \|\widehat{U}^0 - \Lambda_h^n U_h^0\|_X + \|\widehat{U}_h^0 - \Lambda_h^n U_h^0\|_X \leqslant$$

$$\leqslant C\left(n\, h^\theta + \frac{q^n}{1-q}\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} + \frac{M q_h^n}{1-q_h}\|\Lambda_h U_h^0 - U_h^0\|_X.$$

Since

$$\|\Lambda_h U_h^0 - U_h^0\|_X \leqslant C\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}$$

the second estimate from Theorem 3 follows and the proof ends.

Semi-discrete equation with viscosity

We consider the following semi-discrete scheme for (6) with vanishing viscosity

$$\begin{cases} \ddot{u}_h(t) + A_h u_h(t) + B_h B_h^* \dot{u}_h(t) + \gamma h^\eta A_h \dot{u}_h(t) = f_h(t), & (t > 0) \\ u_h(0) = u_{0h}, \ \dot{u}_h(0) = u_{1h}, \end{cases} \tag{25}$$

where $f_h \in \mathcal{C}([0,\infty); V_h)$ and $(u_{0h}, u_{1h}) \in V_h^2$.

Proposition 3

Let $(w_0, w_1) \in X_1$, $(w_h^0, w_h^1) = (\pi_h w^0, \pi_h w^1)$, $f_h(t) = \pi_h f(t)$, $\gamma > 0$ and $\eta \geqslant \theta$. If w and w_h are the solutions of (6) and (25), respectively, then there exist $K_0,\ K_1 > 0$ such that, for each $t \in (0,T)$,

$$\|(w,\dot{w})(t) - (w_h,\dot{w}_h)(t)\|_X \leqslant (K_0 + K_1 T) h^\theta \left(\|(w_0,w_1)\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right). \tag{26}$$

Approximation of the periodic solutions with viscosity

Theorem 4

Let $\widehat{U}^0$ be the unique fixed point of Λ given by Theorem 1 and $\widehat{U}_h^0$ be the unique fixed point of $\Lambda_{h\gamma}$ from Theorem 2. If $\gamma > 0$ and $\eta \geqslant \theta$, there exists a constant $C > 0$ such that, for each $U^0 \in X_1$, $n \geqslant 1$ and $h < h^*$, the following estimates hold

$$\|\widehat{U}^0 - \Lambda_{h\gamma}^n \Pi_h U^0\|_X \leqslant C \left(h^\theta + \frac{q^n}{1-q} \right) \left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right).$$

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant C \left(h^\theta + \frac{q^n}{1-q} + \frac{r^n}{1-r} \right) \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}.$$

Proof of Theorem 4

Let $U^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in X_1$, $U_h^0 = \Pi_h U^0$ and $n \geqslant 1$. We deduce that

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant \|\widehat{U}^0 - \Lambda_{h\gamma}^n U_h^0\|_X + \|\widehat{U}_h^0 - \Lambda_{h\gamma}^n U_h^0\|_X \leqslant$$

$$\leqslant C\left(n\,h^\theta + \frac{q^n}{1-q}\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} + \frac{M q_h^n}{1-q_h}\|\Lambda_{h\gamma}U_h^0 - U_h^0\|_X.$$

Since

$$\|\Lambda_{h\gamma}U_h^0 - U_h^0\|_X \leqslant C\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}$$

estimate (3) follows and the proof ends.

Proof of Theorem 4

Let $U^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in X_1$, $U_h^0 = \Pi_h U^0$ and $n \geqslant 1$. We deduce that

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant \|\widehat{U}^0 - \Lambda_{h\gamma}^n U_h^0\|_X + \|\widehat{U}_h^0 - \Lambda_{h\gamma}^n U_h^0\|_X \leqslant$$

$$\leqslant C\left(h^\theta + \frac{q^n}{1-q}\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} + \frac{Mr^n}{1-r}\|\Lambda_{h\gamma}U_h^0 - U_h^0\|_X.$$

Since

$$\|\Lambda_{h\gamma}U_h^0 - U_h^0\|_X \leqslant C\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}$$

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Does there exist a "bad guy"?

Does there exist a periodic $f_h \in L^1_{loc}(0, \infty; V_h)$ for which the approximation process fails to converge without numerical viscosity?

- The failure of convergence is equivalent to the unboundedness of the initial data $(\widehat{U}_h^0)_{h>0}$ in $X_{\frac{1}{2}}$:

$$\widehat{U}_h^0 = \sum_{1 \leqslant n \leqslant N(h)} a_{hn} \left(i \frac{2\pi n}{T} - \mathbb{A}_h \right)^{-1} \begin{bmatrix} 0 \\ \varphi_{hn} \end{bmatrix}. \tag{27}$$

- A necessary condition for the non convergence would be the unboundedness of $\left\| \left(i \frac{2\pi n}{T} - \mathbb{A}_h \right)^{-1} \right\|$ as n goes to infinity and h tends to zero.

- From the decay of the discrete energy and the resolvent estimates we have

$$\frac{M}{\omega(h)} \geqslant \left\| \left(i\,\frac{2\pi n}{T} - \mathbb{A}_h \right)^{-1} \right\| \geqslant \frac{1}{\min_{\lambda_h \in \sigma(\mathbb{A}_h)} \left| i\,\frac{2\pi n}{T} - \lambda_h \right|}.$$

  A necessary condition for the failure of the convergence process is a kind of "asymptotic resonance phenomenon" expressed by the fact that the distance between the spectrum of $\mathbb{A}_h$ and the set $\mathcal{E} = \left\{ i\,\frac{2\pi n}{T} \,|\, n \geqslant 1 \right\}$ tends to zero as h goes to zero.

- The resonance phenomenon depends, for instance, on the values of the period T and on the velocity at which the high eigenvalues λ_h approach the imaginary axis. This could explain why it is difficult to prove theoretically that there are cases in which the convergence fails and almost impossible to detect them numerically.

Monochromatic periodic terms - a "good guy"

We consider the following particular function f

$$f(t,x) = e^{i\varsigma t} g(x), \tag{28}$$

where $\varsigma \in \mathbb{R}$ and $g \in H$.

Evidently, functions of the form (28) are periodic of period $T = \frac{2\pi}{\varsigma}$ and are usually called *monochromatic*. They appear in many important applications including acoustic, electromagnetic and geophysical wave propagation.

Approximations with monochromatic terms

Theorem 5

Let $\varsigma \in \mathbb{R}$, $T = \frac{2\pi}{\varsigma}$ *and* $g \in H_{\frac{1}{2}}$ *be given and let* $f \in W^{1,1}(0,T;H_{\frac{1}{2}})$ *be defined by* (28)*. By taking* $f_h(t) = e^{i\varsigma t}\pi_h g$*, let* $\widehat{U}^0$ *and* $\widehat{U}_h^0$ *be the unique fixed points of* Λ *and* Λ_h *given by Theorems 1 and 2, respectively. Then there exist* $h_1 > 0$ *and* $K > 0$*, such that for* $h < h_1$ *and* $n \geqslant 1$,

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant K\left(nh^\theta + \frac{q^n}{1-q} + \frac{r^n}{1-r}\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}.$$

A Lemma

Lemma 6

Let $\varsigma \in \mathbb{R}$ be given. There exists $h_0 > 0$ with the property that, for every $h < h_0$, there exist two subspaces W_h^1 and W_h^2 of V_h with

1. *V_h may be written as*

$$V_h = W_h^1 \oplus W_h^2 \tag{29}$$

2. *There exist two positive constants M_1 and ω_1, independent of h, such that*

$$\|\mathbb{S}_h(t)U_h^0\|_X^2 \leqslant M_1 e^{-\omega_1 t}\|U_h^0\|_X \qquad (t \geqslant 0,\ U_h^0 \in W_h^1 \times W_h^1) \tag{30}$$

3. *There exists a constant $C > 0$, independent of h, such that*

$$\left\|(i\,\varsigma\, I - \mathbb{A}_h)^{-1}U_h^0\right\|_X \leqslant Ch^\theta\|U_h^0\|_X \qquad (U_h^0 \in W_h^2 \times W_h^2). \tag{31}$$

Proof of the Lemma

Let $(\varphi_{hn})_{1 \leqslant n \leqslant N(h)}$ and $(\lambda_{hn})_{1 \leqslant n \leqslant N(h)}$ are the sets of eigenvectors normalized in V_h and eigenvalues of the operator $A_h^{\frac{1}{2}}$, respectively. Since $A_h^{\frac{1}{2}}$ is self-adjoint, $(\varphi_{hn})_{1 \leqslant n \leqslant N(h)}$ forms an orthonormal basis in V_h. We define the following two subspace of V_h

$$W_h^1 = \text{Span}\left\{\varphi_{hn} \mid \lambda_{hn} \leqslant \frac{\delta}{h^\theta}\right\}, \qquad W_h^2 = \left[W_h^1\right]^\perp, \tag{32}$$

where $\delta > 0$ is a sufficiently small number to be chosen latter on.

- We have that $V_h = W_h^1 \oplus W_h^2$.
- The third property from Lemma 6 is a consequence of the fact that in W_h^2 we have only high frequencies.

Proof of the Lemma

- The second property from Lemma 6 is a consequence of a result of uniform observability for the low frequencies: there exist positive constants h_0, δ and k_T such that the following inequality holds for any $h < h_0$ and $(w_{0h}, w_{1h}) \in (W_h^1)^2$

$$\int_0^T \|B_h^* \dot{w}_h(t)\|_U^2 \, \mathrm{d}t \geqslant k_T \left(\|A_h^{\frac{1}{2}} w_{0h}\|^2 + \|w_{1h}\|^2 \right), \quad (33)$$

where w_h is the solution of the homogeneous equation

$$\begin{cases} \ddot{w}_h(t) + A_h w_h(t) = 0 \\ w_h(0) = w_{0h}, \qquad \dot{w}_h(0) = w_{1h}. \end{cases}$$

Ervedoza, Sylvain: Spectral conditions for admissibility and observability of wave systems: applications to finite element schemes, Numerische Mathematik, 113 (2009), 377-415.

Proof of Theorem 5

Let $U^0 \in X_1$, $U_h^0 = \Pi_h U^0 \in (W_h^1)^2$ and $n \geqslant 1$. We have that

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant \|\widehat{U}^0 - \Lambda^n U^0\|_X + \|\Lambda^n U^0 - \Lambda_h^n U_h^0\|_X + \|\widehat{U}_h^0 - \Lambda_h^n U_h^0\|_X \leqslant$$

$$\leqslant \frac{Mq^n}{1-q}\|\Lambda U^0 - U^0\|_X + Cn\, h^\theta \,\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} + \|\widehat{U}_h^0 - \Lambda_h^n U_h^0\|_X;$$

If $G_h = \begin{bmatrix} 0 \\ \pi_h g \end{bmatrix}$ then $\widehat{U}_h^0$ is a solution of the equation

$$(i\varsigma - \mathbb{A}_h)\widehat{U}_h^0 = G_h. \tag{34}$$

$$\|\widehat{U}_h^0 - \Lambda_h^n U_h^0\|_X = \|\mathbb{S}_h(nT)(U_h^0 - \widehat{U}_h^0)\|_X \leqslant \left\|\mathbb{S}_h(nT)(U_h^0)\right\|_X +$$

$$+ \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h^1)\right\|_X + \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h^2)\right\|_X \leqslant$$

$$\leqslant C\left(e^{-\omega_1 nT} + e^{-\omega_1 nT} + h^\theta\right).$$

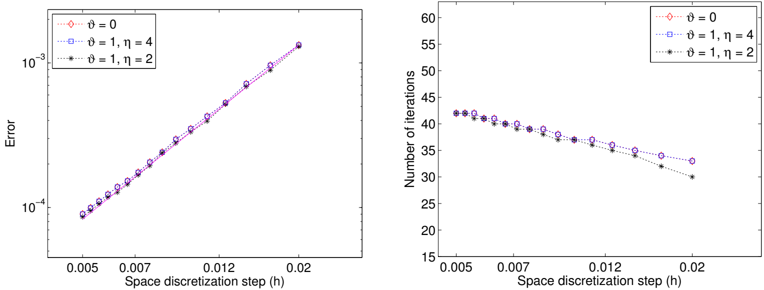
Numerical simulations

$$\begin{cases} \ddot{w}(t,x) - \dfrac{\partial^2 w}{\partial x^2}(t,x) + a(x)\dot{w}(t,x) = f(t,x), \ (t,x) \in (0,\infty) \times (0,1), \\ w(t,0) = w(t,1) = 0, \ t \in (0,\infty) \end{cases}$$

where $a : [0,1] \to \mathbb{R}$ is a nonnegative regular function which is strictly positive in $\omega = (0.2, 0.8)$. The periodic source f is

$$\begin{aligned} f(t,x) =& \alpha t(T-t)\left(6(T-t)^2 - 18t(T-t) + 6t^2\right) x^3(1-x)^3 \\ &- \alpha\left(1 + t^3(T-t)^3 x(1-x)\right)\left(6(1-x)^2 - 18x(1-x) + 6x^2\right) \\ &+ \alpha 3t^2(T-t)^2(T-2t)a(x)x^3(1-x)^3, \end{aligned} \tag{35}$$

for $x \in (0,1)$ and $t \in (0,T)$ and being extended by periodicity to $(0,\infty)$. The corresponding periodic solution is $w(t,x) = \alpha\left(1 + t^3(T-t)^3\right)x^3(1-x)^3$ and the fixed point of the operator Λ is $\widehat{U}^0 = \begin{bmatrix} \alpha x^3(1-x)^3 \\ 0 \end{bmatrix}$.



(a) (b)

Figure: (a) Error for a fixed period $T = 1.5$ and for values of h distributed between $1/200$ and $1/50$. (b) The number of iterations necessary to achieve a precision $\epsilon = h^3$ in the fixed point algorithm for $T = 1.5$ and values of h distributed between $1/200$ and $1/30$.

Let $\Omega = (0,1) \times (0,1)$ be the unit square.

$$\begin{cases} \ddot{w}(t,x) - \Delta w(t,x) + a(x)\dot{w}(t,x) = f(t,x), & (t,x) \in (0,\infty) \times \Omega \\ w(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega \end{cases}$$

where $a \in \mathcal{C}^1(\Omega)$ is a nonnegative function which is strictly positive in an open and non-empty subdomain $\omega \subset \Omega$.

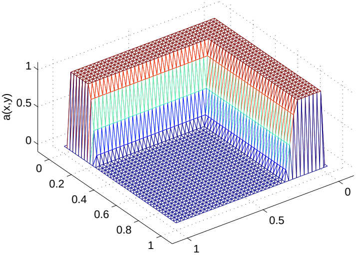


Figure: The function $a \in \mathcal{C}^1(\Omega)$.

The numerical test considered in this case consists in approaching the periodic solution (i.e. the corresponding initial data) associated to the periodic function

$$\begin{aligned} f(t,x,y) =& \alpha\big(6t(T-t)^3 - 18t^2(T-t)^2 + 6t^3(T-t)\big)x^3(1-x)^3y^3(1-y)^3 \\ &- \alpha\left(1+t^3(T-t)^3\right)\big(6x(1-x)^3 - 18x^2(1-x)^2 + 6x^3(1-x)\big)y^3(1-y)^3 \\ &- \alpha\left(1+t^3(T-t)^3\right)\big(6y(1-y)^3 - 18y^2(1-y)^2 + 6y^3(1-y)\big)x^3(1-x)^3 \\ &+ \alpha\big(3t^2(T-t)^3 - 3t^3(T-t)^2\big)a(x,y)x^3(1-x)^3y^3(1-y)^3, \end{aligned}$$

where $\alpha > 0$ is such that $\max\{f(t,x,y) \mid (t,x,y) \in (0,T)\times(0,1)\times(0,1)\} = 1$ and $T = 0.6$. It is easy to see that the periodic solution associated to this function is

$$w(t,x,y) = \alpha\left(1+t^3(T-t)^3\right)x^3(1-x)^3y^3(1-y)^3.$$

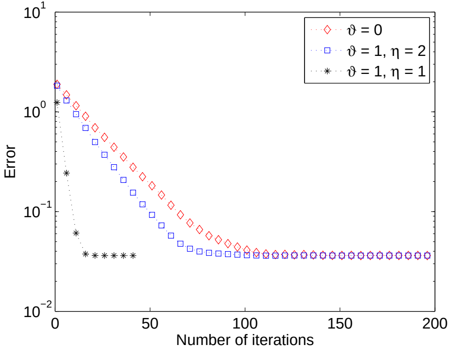


Figure: (a) Evolution of the error in the fixed point algorithm as a function of the iteration's number.

Thank you very much!

Merci beaucoup de votre attention!