Boundary layers and controllability: example with viscous Burgers' equation

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General setting of controllability with boundary layers

Take $T \ll 1$ and $|y_0| \gg 1$.

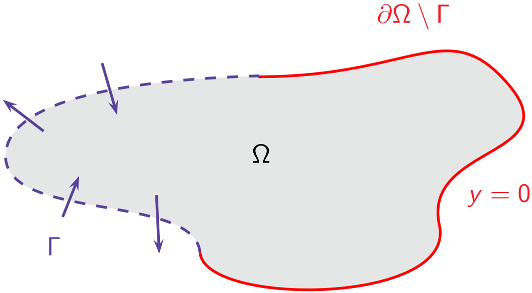

Figure: In such a setting, boundary layers can arise near $\partial\Omega \setminus \Gamma$ while trying to control the fluid inside the domain.

What is a boundary layer?

When $\mathrm{Re} \gg 1$, the fluid's behavior is mostly hyperbolic inside the domain. However...

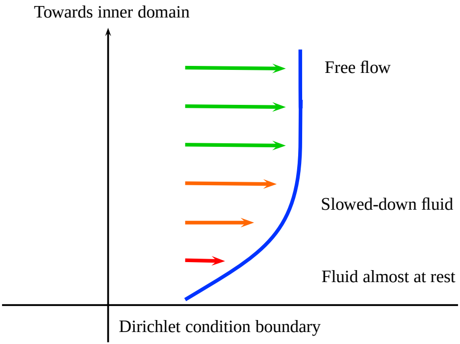

Figure: A boundary layer is a very skinny fluid layer near a boundary where the fluid's behavior is mainly governed by viscous effects and the proximity of the boundary condition.

Our toy model for 1-D boundary layer controllability

Let $y_0 \in L^2(0,1)$ be a given initial data. We consider:

$$\begin{cases} y_t + yy_x - y_{xx} = u(t) & \text{in} \quad (0,T) \times (0,1), \\ y(t,0) = v(t) & \text{in} \quad (0,T), \\ y(t,1) = 0 & \text{in} \quad (0,T), \\ y(0,x) = y_0(x) & \text{in} \quad (0,1). \end{cases} \tag{1}$$

For any controls $u \in L^2(0,T)$ and $v \in H^{1/4}(0,T)$, this system is well-posed in $L^2((0,T); H^1(0,1)) \cap \mathcal{C}^0([0,T]; L^2(0,1))$.

Previous results: two boundary controls

Consider a Burgers' equation with controls acting on both end-points:

$$\begin{cases} y_t + yy_x - y_{xx} = 0 & \text{in} \quad (0,T)\times(0,1), \\ y(t,0) = v(t) & \text{in} \quad (0,T), \\ y(t,1) = w(t) & \text{in} \quad (0,T), \\ y(0,x) = y_0(x) & \text{in} \quad (0,1). \end{cases} \tag{2}$$

Theorem (Guerrero, Imanuvilov, 2007, [2])

System (2) *is not small-time globally null controllable.*

Proof by contradiction, using Cole-Hopf transform and technical lemmas on monotonic behaviors of the heat equation.

Previous results: all three controls

Consider an additionnal inner scalar control:

$$\begin{cases} y_t + yy_x - y_{xx} &= u(t) & \text{in} \quad (0,T)\times(0,1), \\ y(t,0) &= v(t) & \text{in} \quad (0,T), \\ y(t,1) &= w(t) & \text{in} \quad (0,T), \\ y(0,x) &= y_0(x) & \text{in} \quad (0,1). \end{cases} \tag{3}$$

Theorem (Chapouly, 2009, [1])

System (3) is small-time globally null controllable.

Proof relies on the null controllability of the associated inviscid system (proved with Coron's return method), an appropriate scaling and smoothing properties.

Our goal: global null controllability

We want to prove small-time global exact null controllability for our system

$$\begin{cases} y_t + yy_x - y_{xx} &= u(t) & \text{in} \quad (0,T)\times(0,1), \\ y(t,0) &= v(t) & \text{in} \quad (0,T), \\ y(t,1) &= 0 & \text{in} \quad (0,T), \\ y(0,x) &= y_0(x) & \text{in} \quad (0,1). \end{cases}$$

We are in a setting where $|y_0| \gg 1$, $T \ll 1$, ie. high Reynolds numbers.

What about inviscid Burgers' equation?

Consider hyperbolic Burgers' equation:

$$y_t + \frac{1}{2}\partial_x y^2 = u(t) \quad \text{and} \quad \text{boundary conditions.}$$

To achieve global null controllability, we apply constant controls! Can you guess how?

Video for the inviscid system

Correct hyperbolic system

When ε goes to 0, the exact limit-system is:

$$\begin{cases} \bar{y}_t + \frac{1}{2}(\bar{y}^2)_x & = \bar{u}(t) & \text{in} \quad (0,T)\times(0,1), \\ \bar{y}(t,0) & \in E(\bar{v}(t)) & \text{in} \quad (0,T), \\ \bar{y}(t,1) & \geq 0 & \text{in} \quad (0,T), \\ \bar{y}(0,x) & = \bar{y}_0(x) & \text{in} \quad (0,1), \end{cases} \tag{4}$$

where

$$E(\alpha) = \begin{cases}]-\infty;0] & \text{if} \quad \alpha \leq 0, \\ ]-\infty;-\alpha] \cup \{\alpha\} & \text{if} \quad \alpha > 0. \end{cases}$$

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Theorem

System (4) is small-time globally exactly null-controllable.

Can we do likewise for the viscous system?

Now we want to consider $\varepsilon > 0$. What would we get if we used the same controls as those that we have computed for the inviscid case?

Using $v(t)$ to reach a steady-state

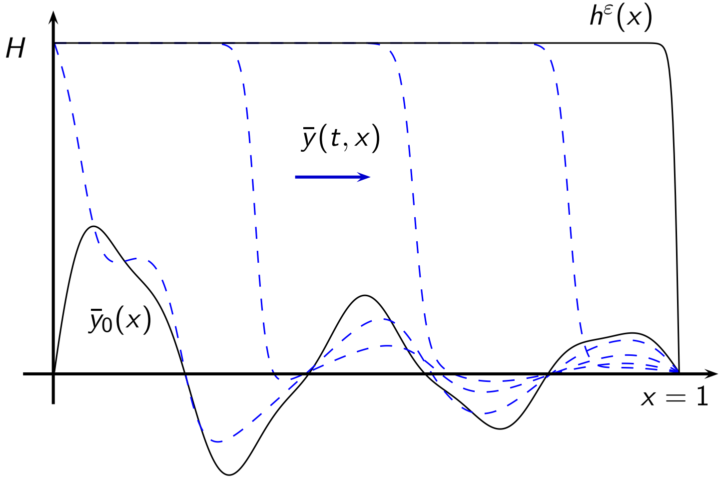


Figure: After a sufficient time (given by the hyperbolic dynamics), we have: $y(t,x) \approx h^{\varepsilon}(x)$

Going back down with $u(t)$

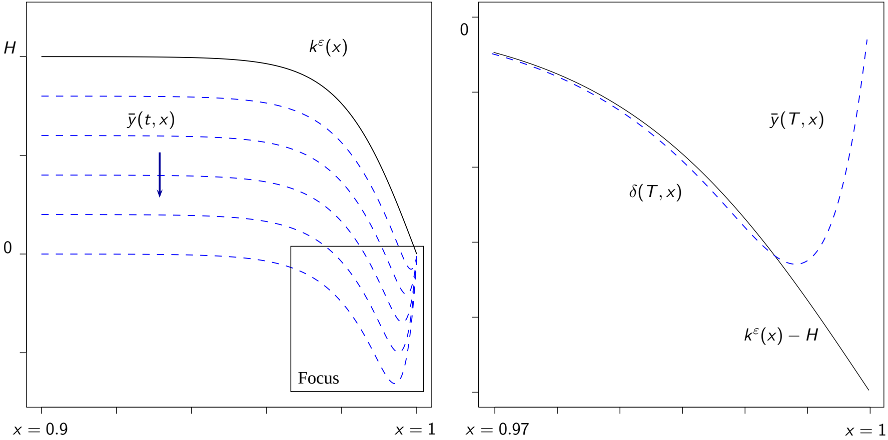
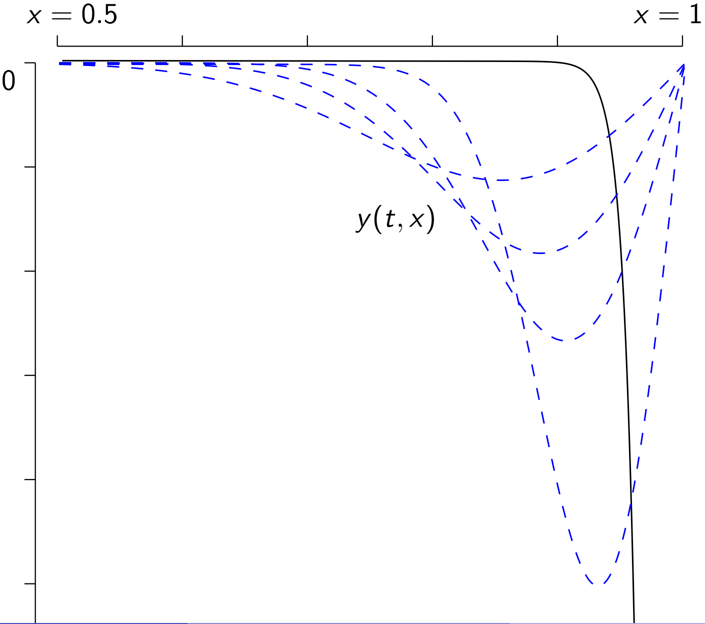


Figure: This action creates a boundary residue Φ^ε

Smoothing effects: no more control!



Putting it all together

We apply $v = H, u = 0$, then $v = 0, u < 0$, then $v = u = 0$.
Video: Viscous control

Proving the dissipation

Step 1: Compute the initial boundary residue $\Phi^\varepsilon(x)$

We want to prove that the video provides the correct result: the boundary layer residue almost vanishes at time T thanks to smoothing effects.

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Thus, Φ^ε converges to a dirac mass when $\varepsilon \to 0$. We want to compute its evolution under the dynamic: $\phi_t + \phi\phi_x - \phi_{xx} = 0$.

Proving the dissipation

Step 2: Perform a Cole-Hopf transform on the system

Introduce z such that:

$$\phi = -2\frac{z_x}{z} \quad \text{and} \quad z(t,x) = \exp\left(-\frac{1}{2}\int_0^x \phi(t,s)\mathrm{d}s\right).$$

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The evolution equation $\phi_t + \phi\phi_x - \phi_{xx} = 0$ becomes:

$$\begin{cases} z_t - z_{xx} &= 0 \quad \text{in} \quad (0,T)\times(0,1), \\ z_x(t,0) = z_x(t,1) &= 0 \quad \text{in} \quad (0,T), \\ z(0,x) &= \text{explicit}(\varepsilon, x) \quad \text{in} \quad (0,1). \end{cases} \tag{5}$$

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Step 3: Use Fourier decomposition

We use the Fourier basis of $L^2(0,1)$ given by $e_n(x) = \sqrt{2}\sin(n\pi x)$.

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&\lesssim \sum_{n\geq 1} \frac{\varepsilon^2 n^2 \pi^2 H^{-2}}{(1+\varepsilon^2 n^2 \pi^2 H^{-2})^2} e^{-2n^2\pi^2 T}
\end{aligned}
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\end{aligned}
$$

Thus, for any $\eta > 0$, if we split the sum at rank $N = \lfloor \varepsilon^{-\eta} \rfloor$ we prove:

$$\|\phi(T,\cdot)\|_{L^2}^2 = \mathcal{O}\left(\varepsilon^{2-3\eta}\right).$$

Summary: main result

System (1) is globally small-time null controllable despite the boundary layer implied by the condition $y(t,1) = 0$.

Theorem

For any $T > 0$, and any $y_0 \in L^2(0,1)$, there exists $u \in L^\infty(0,T)$ and $v \in H^{1/4}(0,T)$ such that the associated solution to system (1) is null at time T.

The proof given in [3] relies on:

- Appropriate scaling to consider small-viscosity instead of small-time.
- Rigorous analysis of the hyperbolic inviscid system.
- Estimates on the creation of the boundary layer.
- Estimates on the dissipation of the boundary layer.
- Small-time local null controllability theorem.

Perspectives: 2-D boundary layers?

Find a 2-D toy model to tackle harder boundary layers. Key difficulties:

- Physical phenomenons are more complex (boundary layers can become unstable or separate from the boundary and enter *inside* the domain).
- Hence, most mathematical models are ill-posed (eg. Prandtl system).
- The geometry plays an important role.

Perspectives: Single control Burgers' system

Is it possible to extend our result using only $u(t)$ and no boundary control?

$$\begin{cases} y_t + yy_x - y_{xx} &= u(t) \quad \text{in} \quad (0,T)\times(0,1), \\ y(t,0) = y(t,1) &= 0 \quad \text{in} \quad (0,T), \\ y(0,x) &= y_0(x) \quad \text{in} \quad (0,1). \end{cases} \tag{6}$$

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Our guess

Even local small-time null controllability does not hold for system (6) because of its *quadratic* structure. A toy model is $y = a - b$, where:

$$\begin{cases} a_t - a_{xx} = u(t), \\ b_t - b_{xx} = aa_x. \end{cases}$$

Thank you for your attention!

Results from the following references were mentionned:

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