Some new results for the controllability of waves equations

J. Le Rousseau (1), P. Terpolilli (2), Gilles Lebeau (3), E.Trélat (4)

(1) Université d'Orléans,
(2) Centre de recherches Total,
(3) Université de Nice,
(4) Université Pierre et Marie Curie.
lebeau@unice.fr

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Outline

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Moving control

Let Ω be a connected bounded domain with smooth boundary, $T > 0$, and $Q =]0, T[\times\Omega$. Let ω be an open subset in Q.
Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. We look for the control problem, with a control $g \in L^2(\omega)$

$$\begin{aligned} &(\partial_t^2 - \triangle)u = \mathbf{1}_\omega g \text{ in } Q, \quad u|_{\partial\Omega} = 0 \\ &u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \end{aligned} \tag{1.1}$$

Let S_T be the continuous affine map from $L^2(\omega)$ into $H_0^1(\Omega) \times L^2(\Omega)$

$$S_T(g) = (u(T,.), \partial_t u(T,.))$$

where $u(t,x)$ is the solution of (1.1).
By definition, exact controllability means that S_T is a surjective map.

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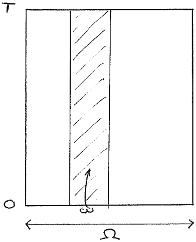
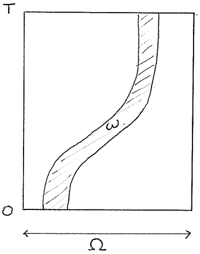
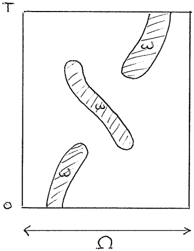


$$(\partial_t^2 - \Delta)u = \mathbb{1}_\omega g$$
$$u|_{\partial\Omega} = 0$$





result

Definition

One says that (ω, T) satisfies the geometric control condition (GCC) iff for any generalized bicharacteristic $t \in \mathbb{R} \mapsto (t, x(t), \tau, \xi(t))$ of the wave operator in Q, there exists $t \in]0, T[$ such that $(t, x(t)) \in \omega$.

Theorem

Assume that (ω, T) satisfies GCC. Then S_T is a surjective map, i.e. the system (1.1) is exactly controllable.

proof

The adjoint system of (1.1) is

$$\begin{aligned} &(\partial_t^2 - \triangle)v = 0 \text{ in } Q, \quad v|_{\partial\Omega} = 0 \\ &v|_{t=0} = v_0 \in L^2(\Omega), \quad \partial_t v|_{t=0} = v_1 \in H^{-1}(\Omega) \end{aligned} \tag{1.2}$$

It is well known that exact controllability is equivalent to the following observability inequality on the adjoint system (1.2): there exists a constant C such that for all $(v_0, v_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ one has

$$\|v_0\|^2_{L^2(\Omega)} + \|v_1\|^2_{H^{-1}(\Omega)} \leq C \int_\omega |v(t,x)|^2 \, dtdx \tag{1.3}$$

For the proof of the observability inequality (1.3), we will follow exactly the well known classical proofs for the non moving case $\omega =]0, T[\times U$, U open in Ω.

proof

The first step is to prove the inequality (for some $s > 0$)

$$\|v_0\|^2_{L^2(\Omega)} + \|v_1\|^2_{H^{-1}(\Omega)} \leq C\Big(\int_\omega |v(t,x)|^2 \, dtdx + \|v_0\|^2_{H^{-s}(\Omega)} + \|v_1\|^2_{H^{-1-s}(\Omega)}\Big) \tag{1.4}$$

For the proof of (1.4), one can use a propagation argument on defect measure, or one can follow the original proof in BLR: Let

$$Y = \{v(t,x) \in H^{-s}(Q), \Box(v) = 0, v|_{\partial\Omega} = 0, v|_\omega \in L^2(\omega)\}$$

$$X = \{v(t,x) \in L^2(Q), \Box(v) = 0, v|_{\partial\Omega} = 0\}$$

One has $X \subset Y$ with continuous embedding. By propagation of the microlocal L^2 regularity on generalized bicharacteristic and the GCC hypothesis, one has $Y = X$, hence (1.4) follows from the Banach theorem.

proof

It remains to deduce the observability inequality (1.3) from the inequality (1.4). Thus, we just have to show that the space of "invisible solutions"

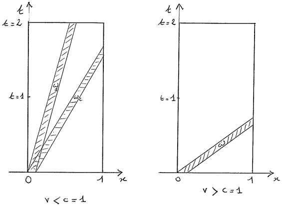
$$N = \{v(t,x) \in L^2(Q), \Box(v) = 0, v|_{\partial\Omega} = 0, v|_\omega = 0\}$$

is reduced to 0. By inequality (1.4), N is a finite dimensional space. Moreover, the Melrose-Sjostrand theorem of propagation of singularities and the GCC hypothesis implies $N \subset C^\infty(\overline{Q})$. Thus N is stable by ∂_t. Then, if $N \neq 0$, there exists in N an eigenfunction of ∂_t, i.e an element of the form $v(t,x) = e^{\lambda t} w(x)$, with

$$\triangle w(x) - \lambda^2 w(x) = 0 \text{ in } \Omega \tag{1.5}$$

By the GCC hypothesis, there exists $t_0 \in]0, T[$ such that $\omega_{t_0} = \omega \cap \{t = t_0\} \neq \emptyset$. One has $w|_{\omega_{t_0}} = 0$, and from (1.5) and the uniqueness theorem for second order elliptic equations, we get $w = 0$. Thus, one has $N = 0$.

examples


t
t = 2
t = 1
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1
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v < c = 1
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t = 2
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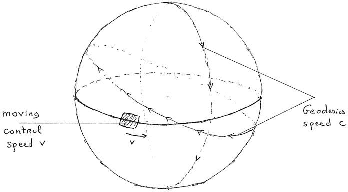
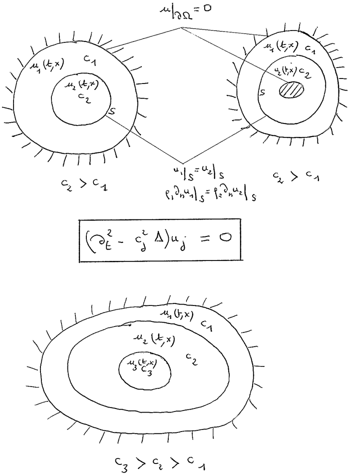
v > c = 1


moving
control
speed v
v
Geodesics
speed c

Outline


$u|_{\partial\Omega} = 0$
$u_1(t,x)$ c_1
$u_2(t,x)$ c_2
S
$c_2 > c_1$
$u_1|_S = u_2|_S$
$\rho_1 \partial_n u_1|_S = \rho_2 \partial_n u_2|_S$
$c_2 > c_1$
$(\partial_t^2 - c_j^2 \Delta) u_j = 0$
$u_1(t,x)$ c_1
$u_2(t,x)$ c_2
$u_3(t,x)$ c_3
$c_3 > c_2 > c_1$

Hypothesis

Let $\Omega_j, 1 \leq j \leq N$ be bounded regular strictly convex domains in $\mathbb{R}^n$ with

$$\overline{\Omega}_{j+1} \subset \Omega_j \quad \forall j \in \{1, ..., N-1\}$$

Set $Q_j =]0, T[\times\Omega_j$. If f is a function, we set $f_j = f|_{Q_j}$. We look at the evolution problem, with $u(t,x) \in C^0([0,T], H_0^1(\Omega_1)) \cap C^1([0,T], L^2(\Omega_1))$

$$\begin{aligned}
&(\partial_t^2 - c_j^2 \triangle) u_j = 0 \text{ in } Q_j \\
&u_1|_{\partial\Omega_1} = 0 \\
&u_{j+1}|_{\partial\Omega_j} = u_j|_{\partial\Omega_j}, \quad \rho_{j+1}\partial_n u_{j+1}|_{\partial\Omega_j} = \rho_j \partial_n u_j|_{\partial\Omega_j} \\
&u(0,x) = f \in H_0^1(\Omega_1), \quad \partial_t u(0,x) = g \in L^2(\Omega_1)
\end{aligned} \tag{2.1}$$

where $\rho_j > 0$ are given constants and the speed $c_j > 0$ are such that

$$c_1 < c_2 < ... < c_N$$

result

Set $b_1 = 1/2$ and for $j = 1, ..., N-1$ $b_{j+1} = b_j \frac{\rho_{j+1} c_j^2}{\rho_j c_{j+1}^2}$.
The system (2.1) is well posed and preserve the total energy

$$E(u) = \sum_{j=1}^{N} b_j \int_{\Omega_j} (|\partial_t u_j(t,x)|^2 + c_j^2 |\nabla u_j(t,x)|^2 \, dx$$

Theorem

There exists $T_0 > 0$ and $C_0 > 0$ such that the following observability inequality holds true for all solution $u(t,x)$ of the system (2.1)

$$\|u_0\|^2_{H_0^1(\Omega_1)} + \|u_1\|^2_{L^2(\Omega_1)} \leq C_0 \int_0^{T_0} \int_{\partial\Omega_1} |\partial_n u_1(t,x)|^2 \, dxdt \qquad (2.2)$$

Remark

We have a geometric description of the minimal time $\inf\{T_0$ *such that* (2.2) *holds true*$\}$. *Observe that in the above result,the observation takes place on* **all the exterior boundary**. *We hope to get in the future a geometric condition on an open subset* $\Gamma \subset \partial\Omega_1$ *of the exterior boundary such that the observability inequality holds true with observation on* Γ.

Remark

If for some $j \in \{1, ..., N-1\}$ *one has* $c_j > c_{j+1}$, *then it is easy to see that the observability inequality* (2.2) *is false, for any time* T_0. *In fact, in the case* $c_j > c_{j+1}$, *there exists "whispering gallery" solutions with energy concentrated on* $\partial\Omega_{j+1}$, *and this contradict the observability inequality. Therefore, the hypothesis*

$$c_1 < c_2 < ... < c_N$$

is a necessary condition for the validity of the observability inequality.

controllability

As an usual consequence of the observability inequality, one gets by duality the following exact controllability result.

Theorem

For any $(f,g) \in L^2(\Omega) \times H^{-1}(\Omega)$, *there exists a control* $h(t,x) \in L^2(]0,T[\times\partial\Omega_1)$ *such that the solution of*

$$\begin{aligned} &(\partial_t^2 - c_j^2 \triangle) u_j = 0 \ \textit{in} \ Q_j \\ &u_1|_{\partial\Omega_1} = h \\ &u_{j+1}|_{\partial\Omega_j} = u_j|_{\partial\Omega_j}, \quad \rho_{j+1}\partial_n u_{j+1}|_{\partial\Omega_j} = \rho_j \partial_n u_j|_{\partial\Omega_j} \\ &u(0,x) = 0, \quad \partial_t u(0,x) = 0 \end{aligned} \tag{2.3}$$

satisfies

$$u(T,x) = f, \quad \partial_t u(T,x) = g \tag{2.4}$$

proof of the observability

As before, the proof split in two parts: First we prove for some $s \in]0,1/2[$

$$\begin{aligned}&\|u_0\|^2_{H^1_0(\Omega_1)} + \|u_1\|^2_{L^2(\Omega_1)} \\ &\leq C\Big(\int_0^{T_0}\int_{\partial\Omega_1} |\partial_n u_1(t,x)|^2\, dxdt + \|u_0\|^2_{H_0^{1-s}(\Omega_1)} + \|u_1\|^2_{H^{-s}(\Omega_1)}\Big)\end{aligned} \tag{2.5}$$

Second, we verify that the space of "invisible" solutions

$$N = \{u \text{ solution of } (2.1),\ \partial_n u|_{\partial\Omega_1} = 0\}$$

is reduced to 0. This second part is the same as before, since by (2.5), N is finite dimensional, and invariant by ∂_t, so we are reduced to an easy uniqueness result for solutions of (2.1) of the form $u(t,x) = e^{\lambda t}w(x)$.

proof of (2.5)

The proof of the penalized observability inequality (2.5) relies on microlocal estimates near the coupling interfaces $\partial\Omega_j, j \geq 2$. We can use defect measures or Sobolev wave front sets up to the boundary. The differences between the two proofs are not essential, and I will indicate the second one. As usual, we set

$$Y = \{u(t,x) \in H^{1-s}(]0, T_0[\times\Omega_1), u \text{ solution of } (2.1), \partial_n v \in L^2(]0, T_0[\times\partial\Omega_1)$$

$$X = \{u(t,x) \in H^1(]0, T_0[\times\Omega_1), u \text{ solution of } (2.1)\}$$

and we have to prove the equality $Y = X$. We will show by induction on $j \in \{1, ..., N\}$ that there exists a sequence of decreasing intervals $I_j \subset]0, T_0[$ of length $T_0 - T_j$, $T_1 < T_2 < ... < T_N$ such that for $T_0 > T_j$

$$Y = Y \cap \{u \in H^1(I_j \times (\Omega_1 \setminus \overline{\Omega}_{j+1}))\}$$

with the convention $\Omega_{N+1} = \emptyset$. Then (2.5) holds true for $T_0 > T_N$.

proof of (2.5)

Let Ω a strictly convex regular domain in $\mathbb{R}^n$. We are interested in (local) solutions near a point $(t_0, x_0) \in \mathbb{R}_t \times \partial\Omega$ of the following transmission problem, with $\rho_\pm > 0$ and $c_+ > c_-$:

$$\begin{aligned}
&(\partial_t^2 - c_+^2 \triangle) u_+ = 0 \text{ in } \mathbb{R}_t \times \Omega \\
&(\partial_t^2 - c_-^2 \triangle) u_- = 0 \text{ in } \mathbb{R}_t \times (\mathbb{R}^n \setminus \overline{\Omega}) \\
&u_+|_{\partial\Omega} = u_-|_{\partial\Omega}, \quad \rho_+ \partial_n u_+|_{\partial\Omega} = \rho_- \partial_n u_-|_{\partial\Omega}
\end{aligned} \tag{2.6}$$

The cotangent bundle $T^*(\mathbb{R}_t \times \partial\Omega)$ split in 5 regions

$$\begin{aligned}
&T^*(\mathbb{R}_t \times \partial\Omega) = \mathcal{H} \cup \mathcal{G}_+ \cup \mathcal{I} \cup \mathcal{G}_- \cup \mathcal{E} \\
&\mathcal{H} = \{(t, y, \tau, \eta),\ \tau^2 > c_+^2 |\eta|^2\} \\
&\mathcal{G}_+ = \{(t, y, \tau, \eta),\ \tau^2 = c_+^2 |\eta|^2\} \\
&\mathcal{I} = \{(t, y, \tau, \eta),\ c_+^2 |\eta|^2 > \tau^2 > c_-^2 |\eta|^2\} \\
&\mathcal{G}_- = \{(t, y, \tau, \eta),\ \tau^2 = c_-^2 |\eta|^2\} \\
&\mathcal{E} = \{(t, y, \tau, \eta),\ \tau^2 < c_-^2 |\eta|^2\}
\end{aligned} \tag{2.7}$$

proof of ([13]2.5)

The set $\mathcal{H}$ is hyperbolic for the two domains Ω and $\mathbb{R}^n \setminus \overline{\Omega}$. For a point $p \in \mathcal{H}$, there exists 4 half rays at p: $\gamma_\pm^{in}, \gamma_\pm^{out}$ with γ_+^* (resp γ_-^*) living above Ω (resp $\mathbb{R}^n \setminus \overline{\Omega}$), and the orientation convention $t(\gamma_\pm^{in}) < t(p) < t(\gamma_\pm^{out})$.

For a point $p \in \mathcal{G}_+$, the two half rays γ_-^* living above $\mathbb{R}^n \setminus \overline{\Omega}$ are still transversal to $\partial\Omega$ at p, but $\gamma_+^{in} \cup \{p\} \cup \gamma_+^{out}$ is a geodesic curve in $\partial\Omega$ with speed c_+.

For a point $p \in \mathcal{I} \cup \mathcal{G}_-$, the two half rays γ_-^* are still here (half straight lines in $\mathbb{R}^n \setminus \overline{\Omega}$ with speed c_- in the case $p \in \mathcal{G}_-$), but γ_+^* do not exists.

Finally, $\mathcal{E}$ is the elliptic region. For a point $p \in \mathcal{E}$, there is no rays at p.

proof of (2.5)

For a local solution of the system (2.6), one can define a wave front set,

$$WF_b(u) \subset T^*(\mathbb{R}_t \times \mathbb{R}^n \setminus \partial\Omega) \cup T^*(\mathbb{R}_t \times \partial\Omega)$$

and also a Sobolev Wave front set $WF_b^s(u)$ with respect to the H^s regularity. We shall use the notation

$$WF_{b,\partial}(u) = WF_b(u) \cap T^*(\mathbb{R}_t \times \partial\Omega)$$

The following theorem is the classical propagation of singularities result:

Theorem

Let u be a solution of the system (2.6). One has $WF_{b,\partial}(u) \cap \mathcal{E} = \emptyset$ and for $p \notin \mathcal{E}$, if the half incoming rays γ_*^{in} are not in $WF_b(u)$ then one has $p \notin WF_{b,\partial}(u)$.

The same result holds true at the level of H^s regularity

proof of ([13] 2.5)

In order to prove the penalized observability inequality ([13] 2.5), one needs an other regularity result, which is used to transfer regularity from $\Omega_j \setminus \overline{\Omega}_{j+1}$ to $\Omega_{j+1} \setminus \overline{\Omega}_{j+2}$.

Theorem

Let u be a solution of the system ([14] 2.6). Then for $p \notin \mathcal{E}$, if $\gamma_-^{in} \cup \gamma_-^{out} \notin WF_b(u)$, then one has $p \notin WF_{b,\partial}(u)$.

The same result holds true at the level of H^s regularity.