High-order elliptic operators: Carleman estimates at boundaries and interfaces

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In collaboration with
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based in parts on works in collaboration with
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Control of Partial Differential Equations,
Conservatoire National des Arts et Métiers, Paris, France
April 2014

Notation

We consider a bounded domain Ω of $\mathbb{R}^d$.
P differential operator of order m (need not be elliptic here)
$\varphi = \varphi(x)$ a smooth weight function to be determined.
$\tau > 0$: large parameter

Form of the estimates away from boundaries

For all K compact in Ω, there exist $C > 0$, $\tau_0 > 0$ such that

$$\sum_{|\beta|<m} \tau^{2(m-|\beta|)-1} \|e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \leq C \|e^{\tau\varphi} P u\|_{L^2}^2, \qquad D = \partial / i \qquad (1)$$

for $u \in \mathscr{C}_c^\infty(\Omega)$, $\mathrm{supp}(u) \subset K$ and $\tau \geq \tau_0$

There are necessary conditions on both P and φ.

There are sufficient conditions on both P and φ.

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Form of the estimates away from boundaries

Example for the Laplace operator, $P = -\Delta$

$$\tau^3 \|e^{\tau\varphi} u\|^2_{L^2} + \tau \|e^{\tau\varphi} \nabla_x u\|^2_{L^2} \leq C \|e^{\tau\varphi} P u\|^2_{L^2},$$

for $u \in \mathscr{C}^\infty_c(\Omega)$, $\operatorname{supp}(u) \subset K$ and $\tau \geq \tau_0$

There are necessary conditions on both P and φ.

There are sufficient conditions on both P and φ.

Some domains of applications

- Unique continuation

  Carleman, Hörmander, Zuily, Alinhac, Robbiano, Lerner, Jerison, Kenig, Tataru, Escauriaza, and many others.

- Inverse problems : Identification of coefficients including stability results

  Bukhgeim, Klibanov, Imanuvilov, Yamamoto, Puel, Uhlmann, Kenig, Isakov, Eller, Dos-Santos Ferreira, Salo, Rosier, Baudouin, Osses, Doubova, Fernández-Cara, Bellassoued, Chouli, Mercado, Benabdallah, Cristofol, Gaitan, Bourgeois, and many others

- Control theory

  Lebeau, Robbiano, Fursikov, Imanuvilov, Fernández-Cara, Zuazua, Guerrero, de Teresa, Gonzales-Burgos, Puel, Rosier, Benabdallah, Ammar-Khodja, Miller, Cepa, Ervedoza, Vancostenoble, Martinez, Cannarsa, and many others

A Carleman estimate of the form

$$\sum_{|\beta|<m} \tau^{2(m-|\beta|)-1} \|e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \lesssim \|e^{\tau\varphi} P u\|_{L^2}^2,$$

can be insufficient to achieve some results, especially for inverse problems and unique continuation.

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Some authors introduce a second large parameter

$$\tau^3 \alpha^4 \|\varphi^{3/2} e^{\tau\varphi} u\|^2_{L^2} + \tau \alpha^2 \|\varphi^{1/2} e^{\tau\varphi} \nabla_x u\|^2_{L^2} \lesssim \|e^{\tau\varphi} \Delta u\|^2_{L^2},$$

with $\varphi(x) = \exp(\alpha\psi(x))$ and $\alpha \geq \alpha_0$.

A Carleman estimate of the form

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Some authors introduce a second large parameter

$$\sum_{|\beta|<m} (\tau\alpha)^{2(m-|\beta|)-1} \|\varphi^{m-|\beta|-1/2} e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \lesssim \|e^{\tau\varphi} P u\|_{L^2}^2, \quad \varphi = e^{\alpha\psi}$$

For parabolic/elliptic operators: **Fursikov-Imanuvillov**
Parabolic operators at the boundary: **Imanuvilov-Puel-Yamamoto**
Hyperbolic operators at the boundary: **Bellassoued-Yamamoto**
For second-order operators: **Isakov, Eller, Kim**

Bellassoued-Yamamoto, 10
Result: A carleman estimate for $\partial_t^2 - \Delta_g$ with two large parameters (global estimate, ie, up to the boudary)

Thanks to the second large parameter it yields Carleman estimates for

- the plate equation
- the thermoelasticity plate equations
- thermoelasticity system with residual stress

We consider

- P be a smooth elliptic of order $m = 2\mu$.

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

  with complex valued coefficients.

- $m/2$ linear smooth boundary operators of order less than m

$$B^k = \sum_{|\alpha| \leq \beta_k} b^k_\alpha(x) D^\alpha, \quad k = 1, \dots, \mu = m/2,$$

  with complex-valued coefficients, defined in some neighborhood of $\partial\Omega$.

Consider the elliptic boundary value problem

$$\begin{cases} Pu(x) = f(x), & x \in \Omega, \\ B^k u(x) = g^k(x), & x \in \partial\Omega, \quad k = 1, \dots, \mu. \end{cases}$$

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We wish to obtain an estimate of the form

$$\|e^{\tau\varphi} u\|^2 + |e^{\tau\varphi} T(u)|^2 \lesssim \Big(\|e^{\tau\varphi} P(x,D) u\|^2 + \sum_{k=1}^{\mu} |e^{\tau\varphi} B^k(x,D) u_{|\partial\Omega}|^2\Big),$$

for u supported near a point at the boundary
$T(u)$ is the trace of $(u, D_\nu u, \dots, D_\nu^{m-1} u)$
If we set

- $P_\varphi = e^{\tau\varphi} P(x,D) e^{-\tau\varphi}$;
- $B_\varphi^k = e^{\tau\varphi} B^k(x,D) e^{-\tau\varphi}$;
- $v = e^{\tau\varphi} u$.

then the Carleman estimate reads:

$$\|v\|^2 + |T(v)|^2 \lesssim \Big(\|P_\varphi v\|^2 + \sum_{k=1}^{\mu} |B_\varphi^k v_{|\partial\Omega}|^2\Big),$$

Estimates of this form were obtained by **Tataru**. We give more precise estimates here and include the complex coefficient case.

2nd-order operators at the boundary were precisely treated by **G. Lebeau and L. Robbiano (95, 97)**.

Set

- $P = -\Delta = \sum_j D_j^2$;
- $\partial\Omega = \{x_n = 0\}$ and $\Omega = \{x_n > 0\}$.

We write $x = (x', x_n)$ and $\xi = (\xi', \xi_n)$.

Take $\varphi = \varphi(x_n)$ such that $\varphi' > 0$.
Then

$$\begin{aligned} P_\varphi &= \big(D_{x_n} + i\tau\varphi'(x)\big)^2 + \overbrace{\sum_{1\leq j\leq n-1} D_j^2}^{M^2} \\ &= \big(D_{x_n} + i(\tau\varphi'(x) + M)\big)\big(D_{x_n} + i(\tau\varphi'(x) - M)\big), \end{aligned}$$

where $M = \operatorname{Op}(|\xi'|)$.

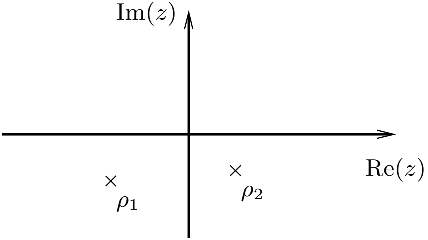
Consider the principal symbol

$$p_\varphi(x,\xi) = p(x, \xi + i\tau\varphi') = \big(\xi_n + i(\tau\varphi'(x) + |\xi'|)\big)\big(\xi_n + i(\tau\varphi'(x) - |\xi|)\big)$$

Principal symbol

$$p_\varphi(x,\xi) = p(x,\xi + i\tau\varphi') = \big(\xi_n + i(\tau\varphi'(x) + |\xi'|)\big)\big(\xi_n + i(\tau\varphi'(x) - |\xi|)\big)$$
$$= \big(\xi_n - \rho_1\big)\big(\xi_n - \rho_2\big)$$

In the low-frequency regime, $|\xi'|$ small,



We have (a microlocal perfect elliptic estimate)

$$\|v\|_{2,\tau} + |T(v)|_{1,1/2,\tau} \lesssim \|P_\varphi v\|_{L^2} \quad (+\cdots)$$

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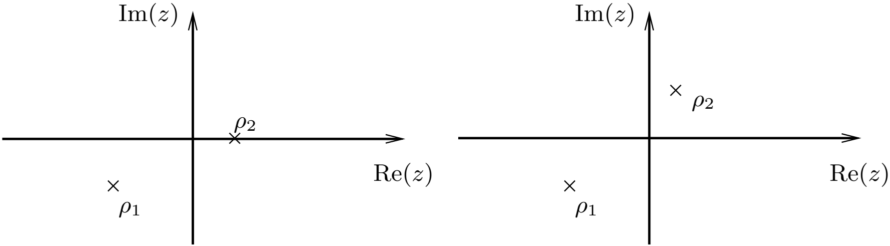
In the high-frequency regime, $|\xi'|$ large,



We have

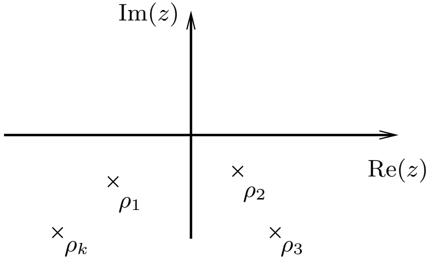
$$\tau^{-1/2}\|v\|_{2,\tau} + |T(v)|_{1,1/2,\tau} \lesssim \|P_\varphi v\|_{L^2} + \text{ boundary norm } \quad (+\cdots)$$

The boundary norm can be of Dirichlet, Neumann, Robin type...

Principal symbol

$$\ell_\varphi(x,\xi) = p(x,\xi + i\tau\varphi') = \prod_{j=1}^{k} \big(\xi_n - \rho_j(x,\tau,\xi')\big)$$

If all the roots have a negative imaginary part,



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$$\|v\|_{k,\tau} + |T(v)|_{k-1,1/2,\tau} \lesssim \|\ell_\varphi v\|_{L^2} \quad (+\cdots)$$

Ideas of the proof: Write $\ell_\varphi = \ell_1 + i\ell_2$, ℓ_1 and ℓ_2 both self adjoint.

$$\|\ell_1 v\|_{L^2}^2 + \|\ell_2 v\|_{L^2}^2 \geq C\|v\|_{k,\tau}^2 - C'|T(v)|_{k-1,\frac{1}{2},\tau}^2$$

(the roots of l_1 and l_2 are real and distinct)

With a generalized green formula we have

$$2(\ell_1 v, \ell_2 v)_{L^2}^2 \geq \mathscr{B}(v) - C\|v\|_{k,-\frac{1}{2},\tau}^2$$

The position of the roots gives

$$\mathscr{B}(v) \gtrsim |T(v)|_{k-1,\frac{1}{2},\tau}^2$$

Set $\varrho' = (x, \xi', \tau)$
We have

$$p_\varphi(\varrho', \xi_n) = \prod_{j=1}^{m} \big(\xi_n - \rho_j(\varrho')\big) = p_\varphi^+(\varrho', \xi_n) p_\varphi^-(\varrho', \xi_n) p_\varphi^0(\varrho', \xi_n),$$

with

$$p_\varphi^\pm(\varrho', \xi_n) = \prod_{\pm \operatorname{Im} \rho_j > 0} (\xi_n - \rho_j), \quad p_\varphi^0(\varrho', \xi_n) = \prod_{\operatorname{Im} \rho_j = 0} (\xi_n - \rho_j).$$

p_φ^- yields a prefect elliptic estimate.

We set

$$\kappa_\varphi(\varrho', \xi_n) = p_\varphi^+(\varrho', \xi_n) p_\varphi^0(\varrho', \xi_n)$$

Boundary operators: $B^k, \quad k = 1, \dots, \mu$

Conjugated operators: $B^k_\varphi = e^{\tau\varphi} B^k e^{-\tau\varphi}$

Principal symbol: $b^k_\varphi(\varrho', \xi_n)$

Strong Lopatinskii condition:

The set $\{b^k_\varphi(\varrho', \xi_n)\}_{k=1,\dots,\mu}$ *is complete modulo* $\kappa_\varphi(\varrho', \xi_n)$ *as polynomials in* ξ_n*.*

For all $f(\xi_n)$ polynomial, there exist $c_1, \dots, c_\mu \in \mathbb{C}$ and $q(\xi_n)$ polynomial such that

$$f(\xi_n) = \sum_{k=1}^{\mu} c_k b^k_\varphi(\varrho', \xi_n) + q(\xi_n)\kappa_\varphi(\varrho', \xi_n)$$

With the strong Lopatinskii condition we obtain

$$|T(v)|_{m-1,1/2,\tau} \lesssim \sum_{k=1}^{\mu} |B_{\varphi}^{k} v_{|x_n=0}|_{m-1/2-\beta_k,\tau} + \|P_{\varphi} v\|_{L^2} \quad (+\cdots)$$

Idea of the proof:
From the strong Lopatinskii condition we have

$$\sum_{k=1}^{\mu} |B_{\varphi}^{k} v_{|x_n=0}|_{m-1/2-\beta_k,\tau} + \sum_{k=0}^{m^- -1} |D_n^k \operatorname{Op}(\kappa_{\varphi}) v_{|x_n=0}|_{m^- -1/2-k,\tau} \gtrsim |T(v)|_{m-1,1/2,\tau}$$

Then use the perfect elliptic estimate for P_{φ}^{-} writing $p_{\varphi} = p_{\varphi}^{-}\kappa_{\varphi}$:

$$\|\operatorname{Op}(\kappa_{\varphi}) v\|_{m^-,\tau} + |T(\operatorname{Op}(\kappa_{\varphi}) v)|_{m^- -1,1/2,\tau} \lesssim \|P_{\varphi} v\|_{L^2} \quad (+\cdots)$$

We write $P_\varphi = A + iB$ with A and B both selfadjoint.

Principal symbols:

$$p_\varphi(\varrho', \xi_n) = \underbrace{a(\varrho', \xi_n)}_{= \operatorname{Re} p_\varphi} + i \underbrace{b(\varrho', \xi_n)}_{= \operatorname{Im} p_\varphi}, \qquad \varrho' = (x, \xi', \tau).$$

Sub-ellipticity property:

$$p_\varphi(\varrho, \xi_n) = 0 \quad \Rightarrow \quad \{a, b\}(\varrho, \xi_n) > 0.$$

Classically we then obtain

$$C\tau^{-1} \|v\|^2_{m,\tau} \leq C'\tau^{-1} \big(\|Av\|^2_{L^2} + \|Bv\|^2_{L^2} + |T(v)|^2_{m-1,1/2,\tau} \big) + \operatorname{Re} \big((Av, iBv) - \mathscr{B}_{a,b}(v) \big)$$

Assuming the Strong Lopatinskii condition and the sub-ellipticity condition

We recall

$$|T(v)|^2_{m-1,1/2,\tau} \lesssim \sum_{k=1}^{\mu} |B^k_\varphi v_{|x_n=0}|^2_{m-1/2-\beta_k,\tau} + \|P_\varphi v\|^2_{L^2} \quad (+\cdots)$$

and

$$C\tau^{-1}\|v\|^2_{m,\tau} \leq C'\tau^{-1}\big(\|Av\|^2_{L^2} + \|Bv\|^2_{L^2} + |T(v)|^2_{m-1,1/2,\tau}\big) + \operatorname{Re}\big((Av, iBv) - \mathscr{B}_{a,b}(v)\big)$$

We have $|\mathscr{B}(v)| \lesssim |T(v)|^2_{m-1,1/2,\tau}$.

Combining the two blue estimates we obtain

$$\tau^{-1}\|v\|^2_{m,\tau} + |T(v)|^2_{m-1,1/2,\tau} \lesssim \|P_\varphi v\|^2_{L^2} + \sum_{k=1}^{\mu} |B^k_\varphi v_{|x_n=0}|^2_{m-1/2-\beta_k,\tau}$$

for τ large.

We have thus obtained

THEOREM (Bellassoued, LR)

Under

- *sub-ellipticity condition,*
- *strong Lopatinskii condition,*

Let $x_0 \in \partial\Omega$. There exist W a neighborhood of x_0, $C > 0$, and $\tau_0 > 0$ such that at the boundary

$$\tau^{-1} \|e^{\tau\varphi} u\|^2_{m,\tau} + |e^{\tau\varphi} T(u)|^2_{m-1,1/2,\tau} \leq C \Big(\|e^{\tau\varphi} P(x,D) u\|^2_{L^2} + \sum_{k=1}^{\mu} |e^{\tau\varphi} B^k(x,D) u_{|\partial\Omega}|^2_{m-1/2-\beta_k,\tau} \Big),$$

for $\tau \geq \tau_0$ and $u = w_{|\Omega}$ with $w \in \mathscr{C}^\infty_c(W)$.

Let $\varphi = e^{\alpha\psi}$. With proper pseudo-differential calculus **[LR12]** we can obtain

THEOREM (Bellassoued, LR)

Under

- *strong pseudo-convexity condition,*
- *strong Lopatinskii condition,*

Let $x_0 \in \partial\Omega$. *There exist* W *a neighborhood of* x_0, $C > 0$, $\tau_0 > 0$, α_0 *such that at the boundary*

$$\|\tilde{\tau}^{-\frac{1}{2}} e^{\tau\varphi} u\|^2_{m,\tilde{\tau}} + |e^{\tau\varphi} T(u)|^2_{m-1,1/2,\tilde{\tau}} \leq C\Big(\|e^{\tau\varphi} P(x,D) u\|^2_{L^2} + \sum_{k=1}^{\mu} |e^{\tau\varphi} B^k(x,D) u_{|\partial\Omega}|^2_{m-1/2-\beta_k,\tilde{\tau}} \Big),$$

for $\tau \geq \tau_0$, $\alpha \geq \alpha_0$ *and* $u = w_{|\Omega}$ *with* $w \in \mathscr{C}^\infty_c(W)$.

Here $\tilde{\tau} = \tau\alpha\varphi$.

Let $\varphi = e^{\alpha\psi}$. With proper pseudo-differential calculus [LR12] we can obtain

THEOREM (Bellassoued, LR)

Under

- *the simple characteristic condition,*
- *strong Lopatinskii condition,*

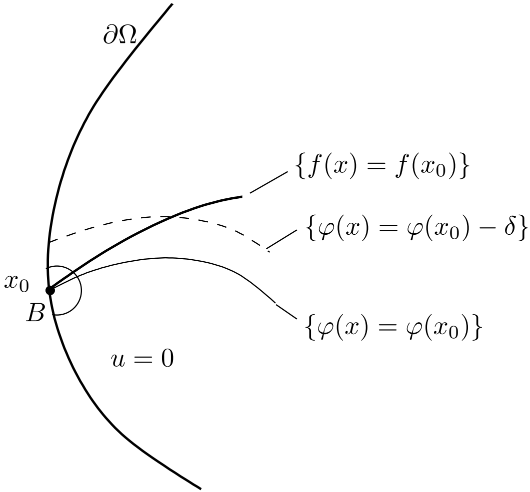
Let $x_0 \in \partial\Omega$. There exist W a neighborhood of x_0, $C > 0$, $\tau_0 > 0$, α_0 such that at the boundary

$$\alpha \|\tilde{\tau}^{-\frac{1}{2}} e^{\tau\varphi} u\|^2_{m,\tilde{\tau}} + |e^{\tau\varphi} T(u)|^2_{m-1,1/2,\tilde{\tau}} \leq C\Big(\|e^{\tau\varphi} P(x,D)u\|^2_{L^2} + \sum_{k=1}^{\mu} |e^{\tau\varphi} B^k(x,D) u_{|\partial\Omega}|^2_{m-1/2-\beta_k,\tilde{\tau}}\Big),$$

for $\tau \geq \tau_0$, $\alpha \geq \alpha_0$ and $u = w_{|\Omega}$ with $w \in \mathscr{C}^\infty_c(W)$.

Here $\tilde{\tau} = \tau\alpha\varphi$.

Application: unique continuation at the boundary under strong pseudo-convexity condition



THEOREM (Bellassoued-LR)

Let $x_0 \in \partial\Omega$, $f \in \mathscr{C}^\infty(\overline{\Omega})$, and V a neighborhood of x_0 such that

1. *the strong pseudo convexity property is fulfilled in V*
2. *the strong Lopatinskii condition holds at x_0*

Assume that $u \in H^m(\Omega)$ satisfies

- $$|Pu(x)| \leq C \sum_{|\alpha| \leq m-1} |D^\alpha u(x)|, \quad a.e.\ in\ V;$$
- *for $k = 1, \dots, \mu$*
  $$|B^k u(x)| \leq C \sum_{|\alpha| \leq \beta_k - 1} |D^\alpha u(x)|, \quad a.e.\ in\ a\ neighborhood\ of\ V \cap \partial\Omega,$$
- *and u vanishes in $\{x \in V;\ f(x) \geq f(x_0)\}$.*

Then u vanishes in a neighborhood of x_0.

In the same setting, let P_1 and P_2 be of order m_1 and m_2 respectively.

If P_1, f, B_1^k, k=1, . . . , $m_1/2$ satisfy

- the strong pseudo convexity property
- the strong Lopatinskii condition holds at x_0

If P_2, f, B_2^k, k=1, . . . , $m_2/2$ satisfy

- the single characteristic property
- the strong Lopatinskii condition holds at x_0

Then a similar unique continuation result holds for $P = P_1 P_2$.

We wish to obtain similar results for the case of transmission problem across an interface.

Setting $\Omega = \Omega_1 \cup S \cup \Omega_2$
S = interface between Ω_1 and Ω_2.

- P_1 and P_2 elliptic operators of order $m_1 = 2\mu_1$ and $m_2 = 2\mu_2$ (possibly different).
- $m_1 + m_2$ linear transmission operators

$$S_k^j = \sum_{|\alpha| \leq \beta_k^j} s_{k,\alpha}^j(x) D^\alpha, \quad k = 1, 2, \quad j = 1, \ldots, m = \mu_1 + \mu_2. \quad (1)$$

We consider the following elliptic transmission problem

$$\begin{cases} P_k u_k = f_k & \text{in } \Omega_k, \ , k = 1, 2 \\ S_1^j u_1 + S_2^j u_2 = g^j, & \text{in } S, \quad j = 1, \ldots, m. \end{cases}$$

in addition with boundary conditions.

The type of estimate that we wish to obtain is of the form

$$\sum_{k=1,2} \left(\tau^{-1} \|e^{\tau\varphi_k} u_k\|^2_{m_k,\tau} + |e^{\tau\varphi_k} T(u_k)|^2_{m_k-1,1/2,\tau}\right)$$

$$\lesssim \Big(\sum_{k=1,2} \|e^{\tau\varphi_k} P_k(x,D) u_k\|^2_{L^2}$$

$$+ \sum_{j=1}^{m} |e^{\tau\varphi_{|S}} (S_1^j(x,D)u_1 + S_2^j(x,D)u_2)_{|S}|^2_{m-1/2-\beta^j,\tau}\Big),$$

2nd-order operators at an interface were treated by **[LR–Robbiano, 10; LR–Lerner 13]**.

Set

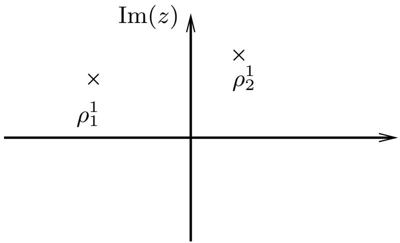
- $P_1 = -\nabla c_1 \nabla; \quad P_2 = -\nabla c_2 \nabla;$
- $S = \{x_n = 0\}$, $\Omega_1 = \{x_n < 0\}$, and $\Omega_2 = \{x_n > 0\}$.

We write $x = (x', x_n)$ and $\xi = (\xi', \xi_n)$.

Take $\varphi = \varphi(x_n)$ such that $\varphi' > 0$.

Then

$$p_{j,\varphi}(x,\xi) = p_j(x, \xi + i\tau\varphi') = \big(\xi_n - \rho_1^j\big)\big(\xi_n - \rho_2^j\big)$$



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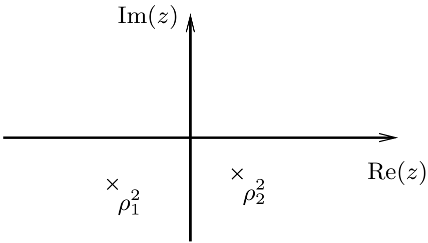
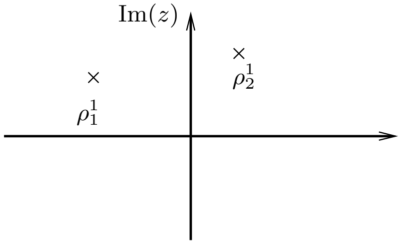


two roots are determined



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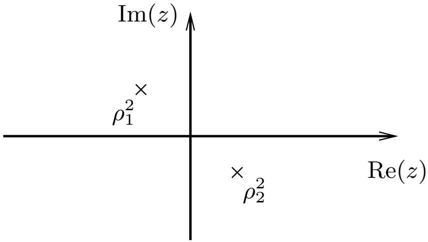
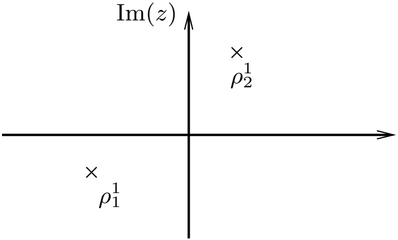
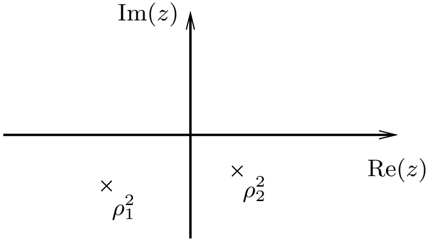


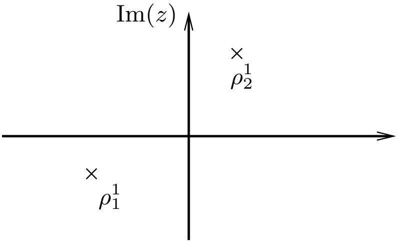
One relation



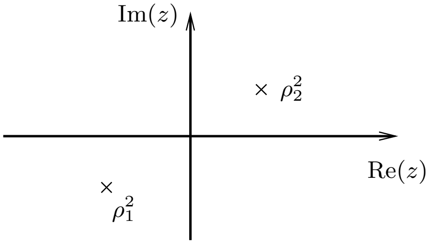
one relation



two roots are dertermined



one relation



one relation

Conditions to always be able to solve for the traces at the interface are put forward in **[LR–Robbiano]**.

These conditions are proven sharp in **[LR–Lerner]** A quasi-mode is constructed otherwise.

These conditions are generalized here for the high-order transmission problem.

we set

- $P_{j,\varphi} = e^{\tau\varphi_j} P_j(x,D) e^{-\tau\varphi_j}$;
- $S^k_{j,\varphi} = e^{\tau\varphi_j} S^k_j(x,D) e^{-\tau\varphi_j}$;
- $v_j = e^{\tau\varphi_j} u_j$.

We have

$$p_{j,\varphi}(\varrho', \xi_n) = \prod_{k=1}^{m_j} \big(\xi_n - \rho^j_k(\varrho')\big) = p^+_{j,\varphi}(\varrho', \xi_n) p^-_{j,\varphi}(\varrho', \xi_n) p^0_{j,\varphi}(\varrho', \xi_n),$$

with

$$p^{\pm}_{j,\varphi}(\varrho', \xi_n) = \prod_{\pm \operatorname{Im} \rho^j_k > 0} (\xi_n - \rho^j_k), \quad p^0_{\varphi}(\varrho', \xi_n) = \prod_{\operatorname{Im} \rho^j_k = 0} (\xi_n - \rho^j_k).$$

We set

$$\kappa_{j,\varphi}(\varrho', \xi_n) = p^+_{j,\varphi}(\varrho', \xi_n) p^0_{j,\varphi}(\varrho', \xi_n)$$

The transmission condition for $\{P_j, S_j^j, \varphi_j,\ j = 1, \ldots, m\}$ reads:

for all pairs of polynomials, $q_j(\xi_n)$, there exist U_j, polynomials, and $c_k \in \mathbb{C}$, $k = 1, \ldots, m = m_1 + m_2$, such that:

$$q_1(\xi_n) = \sum_{k=1}^{m} c_k s_{1,\varphi}^k(\varrho', \xi_n) + U_1(\xi_n)\kappa_{1,\varphi}(\varrho', \xi_n),$$

and

$$q_2(\xi_n) = \sum_{k=1}^{m} c_k s_{2,\varphi}^k(\varrho', \xi_n) + U_2(\xi_n)\kappa_{2,\varphi}(\varrho', \xi_n).$$

The important proposition then becomes

Proposition

Assume that the transmission condition is satisfied then

$$C\big(|T(v_1)|_{m_1-1,1/2,\tau} + |T(v_2)|_{m_2-1,1/2,\tau}\big) \leq \sum_{k=1}^{m} |S^k_{1,\varphi} v_{1|x_n=0} + S^k_{2,\varphi} v_{2|x_n=0}|_{m-1/2-\beta^k,\tau} + \|P_{1,\varphi} v_1\|_{L^2} + \|P_{2,\varphi} v_2\|_{L^2} + (...),$$

for $\tau \geq \tau_0$, $v_1, v_2 \in \mathscr{C}^\infty$.

Then the Carleman estimate follows.

Theorem (LR–Bellassoued)

Let $x_0 \in S$ and let $\varphi \in \mathscr{C}^0(\Omega)$ be such that $\varphi_k = \varphi_{|\Omega_k} \in \mathscr{C}^\infty(\Omega_k)$ for $k = 1, 2$ and such that the pairs $\{P_k, \varphi_k\}$ have the sub-ellipticity property in a neighborhood of x_0 in $\overline{\Omega_k}$. Moreover, assume that $\big\{P_k, \varphi, S_k^j,\ k = 1,2,\ j = 1, \ldots, \mu\big\}$ satisfies the transmission condition at x_0. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and two constants C and $\tau_* > 0$ such that

$$\sum_{k=1,2} \big(\tau^{-1} \| e^{\tau\varphi_k} u_k \|^2_{m_k,\tau} + | e^{\tau\varphi_k} T(u_k) |^2_{m_k-1,1/2,\tau}\big) \\ \leq C \Big(\sum_{k=1,2} \| e^{\tau\varphi_k} P_k(x,D) u_k \|^2_{L^2} \\ + \sum_{j=1}^{m} | e^{\tau\varphi_{|S}} (S_1^j(x,D) u_1 + S_2^j(x,D) u_2)_{|S} |^2_{m-1/2-\beta^j,\tau} \Big),$$

for all $u_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}_c^\infty(W)$ and $\tau \geq \tau_*$.

Version with two large parameters.

Theorem (LR–Bellassoued)

Let $x_0 \in S$ and let $\psi \in \mathscr{C}^0(\Omega)$ be such that $\psi_k = \psi_{|\Omega_k} \in \mathscr{C}^\infty(\Omega_k)$ for $k = 1, 2$ and such that ψ_k have the strong pseudo-convexity property with respect to P_k in a neighborhood of x_0 in $\overline{\Omega_k}$. Moreover, assume that $\{P_k, \psi, S_k^j,\ k = 1, 2,\ j = 1, \ldots, \mu\}$ satisfies the transmission condition at x_0. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\alpha_* > 0$ such that for $\varphi_k = \exp(\alpha\psi_k)$ and $\tilde{\tau}_k = \tau\alpha\varphi_k$:

$$\sum_{k=1,2} \left(\left\| \tilde{\tau}_k^{-1/2} e^{\tau\varphi_k} u_k \right\|^2_{m_k, \tilde{\tau}_k} + \left| e^{\tau\varphi_{|S}} T(u_k) \right|^2_{m_k-1, 1/2, \tilde{\tau}} \right) \leq C \Big(\sum_{k=1,2} \| e^{\tau\varphi_k} P_k(x, D) u_k \|^2_{L^2} + \sum_{j=1}^{m} \left| e^{\tau\varphi_{|S}} \left(S_1^j(x, D) u_{1|S} + S_2^j(x, D) u_{2|S} \right) \right|^2_{m-1/2-\beta^k, \tilde{\tau}} \Big),$$

for all $u_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}_c^\infty(W)$, $\tau \geq \tau_*$ and $\alpha \geq \alpha_*$.

Version with two large parameters.

Theorem (LR–Bellassoued)

Let $x_0 \in S$ and let $\psi \in \mathscr{C}^0(\Omega)$ be such that $\psi_k = \psi_{|\Omega_k} \in \mathscr{C}^\infty(\Omega_k)$ for $k = 1, 2$ and such that ψ_k have the simple characteristic property with respect to P_k in a neighborhood of x_0 in $\overline{\Omega_k}$. Moreover, assume that $\{P_k, \psi, S_k^j,\ k = 1, 2,\ j = 1, \dots, \mu\}$ satisfies the transmission condition at x_0. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\alpha_* > 0$ such that for $\varphi_k = \exp(\alpha\psi_k)$ and $\tilde{\tau}_k = \tau\alpha\varphi_k$:

$$\sum_{k=1,2} \Big(\alpha \big\|\tilde{\tau}_k^{-1/2} e^{\tau\varphi_k} u_k\big\|^2_{m_k,\tilde{\tau}_k} + |e^{\tau\varphi_{|S}} T(u_k)|^2_{m_k-1,1/2,\tilde{\tau}}\Big)$$

$$\leq C\Big(\sum_{k=1,2} \|e^{\tau\varphi_k} P_k(x,D) u_k\|^2_{L^2}$$

$$+ \sum_{j=1}^{m} |e^{\tau\varphi_{|S}} (S_1^j(x,D) u_{1|S} + S_2^j(x,D) u_{2|S})|^2_{m-1/2-\beta^k,\tilde{\tau}}\Big),$$

for all $u_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}_c^\infty(W)$, $\tau \geq \tau_*$ and $\alpha \geq \alpha_*$.

With the above estimates we can deduce unique continuation results near an interface.