On Exact Boundary Controllability of 1-D Degenerate Equations

Mamadou Gueye[1]

[1]Departamento de Matemática, Universidad Técnica Federico Santa María, Chile

Control of PDEs 2014

Conservatoire national des arts et métiers, Paris

One dimensional degenerate equations

✓ Let $T>0$ and $Q:=(0,1)\times(0,T)$. Let $a:[0,1]\mapsto\mathbb{R}$, $a\in\mathcal{C}^0([0,1])\cap\mathcal{C}^1((0,1])$, $a>0$ on $(0,1]$ and $a(0)=0$.

✓ Two simples degenerate equations

$$\begin{cases} y_t-(a(x)y_x)_x=v\mathbf{1}_\omega \quad \text{in } Q, \\ y(1,t)=\rho(t), \quad \begin{cases} y(0,t)=\gamma(t), & \text{weak degeneracy} \\ \{a(x)y_x(x,t)\}_{|x=0}=0, & \text{strong degeneracy} \end{cases} & \text{on } (0,T). \end{cases} \tag{P}$$

$$\begin{cases} y_{tt}-(a(x)y_x)_x=v\mathbf{1}_\omega \quad \text{in } Q, \\ y(1,t)=\rho(t), \quad \begin{cases} y(0,t)=\gamma(t), & \text{weak degeneracy} \\ \{a(x)y_x(x,t)\}_{|x=0}=0, & \text{strong degeneracy} \end{cases} & \text{on } (0,T). \end{cases} \tag{H}$$

✓ For non degenerate equations we refer to [Fattorini, Russell '71] [Imanuvilov-Fursikov, Lebeau-Robbiano '95]...

A class of degenerate heat equations

$$\begin{cases} y_t - (x^\alpha y_x)_x = v\chi_\omega & \text{in } Q, \\ y(1,t) = 0, \quad \begin{cases} y(0,t) = 0, & 0 \le \alpha < 1 \\ \{x^\alpha y_x(x,t)\}_{|x=0} = 0, & 1 \le \alpha \end{cases} & \text{on } (0,T). \end{cases} \tag{P}$$

✓ Controllability for $\alpha \in [0,2)$ [Cannarsa, Vancostenoble, Martinez '08], generalized to diffusions behaving like $xa'(x)/a(x) \to \alpha$, as $x \to 0$.

✓ Non controllability for $\alpha \ge 2$ [S. Micu, E. Zuazua '01] [L. Escauriaza, G. Seregin, V. Sverák '04]. For $\alpha > 2$ set (also for $\alpha = 2$)

$$X := \int_x^1 s^{-\alpha/2} \mathrm{d}s, \quad U(t,X) := x^{\alpha/4} y(x,t), \tag{Liouv}$$

$$U_t - U_{XX} + P(X)U = V\chi_{\tilde{\omega}}, \quad X \in (0,\infty),\ P(X) := \frac{\alpha(3\alpha-4)}{(4+(2\alpha-4)X)^2}$$

✓ Approximate controllability at $\{x = 0\}$ for $\alpha \in [0,1)$ [Cannarsa, Tort, Yamamoto '11].

Transmutation method I

Parabolic evolution equation

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial y}{\partial x}\right) \quad x \in Q, \quad y(x,0) = y^0(x) \quad x \in (0,1). \tag{P}$$

For $L > 0$ let $k_T : (-L, L) \times (0,T) \mapsto \mathbb{R}$. Introduce the integral transform

$$z(x,s) := \int_0^T k_T(s,t) y(x,t)\, \mathrm{d}t.$$

Hyperbolic evolution equation [Petit, Rouchon, '00]

$$\frac{\partial^2 z}{\partial s^2} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial z}{\partial x}\right) \quad (x,s) \in (0,1) \times (-L, L), \tag{H}$$

If we choose $k_T(\cdot,\cdot)$ to be the heat kernel

$$\begin{cases} \partial_t k_T + \partial_{ss} k_T = 0 & (s,t) \in (-L,L) \times (0,T), \\ k_T(s,0) = 0,\ k_T(s,T) = 0 & s \in (-L,L), \\ k_T(0,t) = 0 & t \in (0,T). \end{cases} \tag{Ker}$$

If y is a solution of (P), then z is a solution of (H) with

$$z(x,0) = 0, \quad z_s(x,0) = \int_0^T \partial_s k_T(0,t) y(x,t)\, \mathrm{d}t \quad x \in (0,1).$$

Transmutation method II

Heat process	$\longrightarrow$	Wave process
Observability Heat	$\longleftarrow$	Observability Wave

Let $T > 0$ and $L > 0$. Then, for any $\beta > 2L^2$, there exists a function $k_T = k_T(t,s)$ satisfying (Ker) and such that

$$\partial_s k_T(0,t) = \exp\left(-\beta\left(\frac{1}{t} + \frac{1}{T-t}\right)\right), \quad t \in (0,T).$$

Moreover, for all $\delta \in (0,1)$, k_T satisfies the following estimates

$$|k_T(s,t)| \leq |s| \exp\left(\frac{1}{\min\{t, T-t\}}\left(\frac{s^2}{\delta} - \frac{\beta}{1+\delta}\right)\right),$$

for all $(s,t) \in (-L,L) \times (0,T)$.

A proof of this result can be found in [Ervedoza, Zuazua, '11].

A singular Sturm-Liouville problem

Consider the differential expression defined by

$$-(x^{\alpha}y'(x))' = \lambda y(x), \quad x \in (0,1), \lambda \in \mathbb{R}.$$

✓ Particular case of Bessel differential equation

$$x^2 y'' + axy' + (bx^m + c)y = 0, \quad x \in (0,\infty), \quad m \neq 0.$$

[Kamke '39] exhibits solutions of the form

$$b \neq 0 : \quad y = x^{\frac{1}{2}(1-a)} Z_{\nu}(\kappa^{-1}\sqrt{b}x^{\kappa}), \quad \nu := \frac{1}{m}\sqrt{(1-a)^2 - 4c}, \quad \kappa := \frac{m}{2},$$

where Z_{ν} is any Bessel function J_{ν}, Y_{ν}, $H_{\nu}^{(1)}$, $H_{\nu}^{(2)}$ of order ν.

✓ Endpoint classification $0,1$ in R, LP, LC (LCO, LCNO). Connected to the type of boundary conditions we can impose.

✓ For $\alpha \in (0,1)$ the endpoint 0 is R. The Dirichlet boundary conditions make sense.

Functional settings

✓ For $\alpha \in [0,2)$ set the following weighted Sobolev space

$$H^1_\alpha(0,1) := \left\{ f \in L^2(0,1) \cap H^1_{\text{loc}}(0,1) \mid x^{\alpha/2} f' \in L^2(0,1) \right\}, \qquad \text{(WSob)}$$

$$D_{\max} := \left\{ f,\, x^\alpha f' \in AC_{\text{loc}}(0,1) |\; f,\, (x^\alpha f')' \in L^2(0,1) \right\}.$$

✓ For $\alpha \in [0,1)$ set the following space

$$H^1_{\alpha,0}(0,1) := \left\{ f \in H^1_\alpha(0,1) \mid f(0) = f(1) = 0 \right\}.$$

For $\alpha \in [1,2)$ the functions in $H^1_\alpha(0,1)$ may be eventually unbounded at endpoints.

✓ Define the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1)$

$$\begin{cases} D(\mathcal{A}) := \left\{ \; f \in H^1_{\alpha,0}(0,1) \mid x^\alpha f' \in H^1(0,1) \; \right\}, \\ \forall\, f \in D(\mathcal{A}),\; \mathcal{A}f := -(x^\alpha f')' \,. \end{cases} \qquad \text{(D)}$$

Observability for the degenerate wave equation

Let $(\Phi_n)_{n\in\mathbb{N}^*}$ be the eigenfuntions of $\mathcal{A}$, $(\lambda_n)_{n\in\mathbb{N}^*}$ the associated eigenvalues. This enables us to introduce interpolation spaces

$$\mathbf{H}^s_\alpha := D(\mathcal{A}^{s/2}) = \left\{ f = \sum_{n\geq 1} a_n\Phi_n \,|\, \|f\|_s^2 := \sum_{n\geq 1} |a_n|^2\lambda_n^s < \infty \right\}, \quad s\in\mathbb{R}.$$

Adjoint observability problem

$$\begin{cases} z_{tt} - (x^\alpha z)_x = 0 & (x,t)\in(0,1)\times(0,T), \\ z(0,t) = 0, \quad z(1,t) = 0 & t\in(0,T), \\ z(x,0) = z^0, \quad z_x(x,0) = z^1 & x\in(0,1). \end{cases} \qquad (\mathsf{H})$$

Théorème For $\alpha\in[0,1)$ let $\nu = (1-\alpha)\backslash(2-\alpha)$. There exists $T_\alpha > 0$ such that for every $T > T_\alpha$, the system (H) is observable in the space $\mathbf{H}_\alpha^{\frac{1}{2}(2\nu+1)}\times\mathbf{H}_\alpha^{\frac{1}{2}(2\nu-1)}$ in time T by an observation on $\{x=0\}$ in $L^2(0,T)$, moreover

$$\left\|z^0\right\|^2_{\frac{1+2\nu}{2}} + \left\|z^1\right\|^2_{\frac{-1+2\nu}{2}} \asymp \int_0^T \left\{ x^\alpha \frac{\partial z}{\partial x}(x,t)\right\}^2_{|x=0} \mathrm{d}t\,. \qquad (\mathsf{Obs})$$

Controllability for the degenerate wave equation

Consider the following control problem :

$$\begin{cases} w_{tt} - (x^\alpha w)_x = 0 & (x,t) \in Q, \\ w(0,t) = \theta(t), \quad w(1,t) = 0 & t \in (0,T), \\ w(x,0) = 0, \quad w_x(x,0) = 0 & x \in (0,1). \end{cases} \tag{H}$$

Théorème For $\alpha \in [0,1)$ let $\nu = (1-\alpha)\backslash(2-\alpha)$. Let $T > T_\alpha$. Then, for every

$$(w^0, w^1) \in \mathbf{H}_\alpha^{\frac{1-2\nu}{2}} \times \mathbf{H}_\alpha^{\frac{-1-2\nu}{2}},$$

there exists a control $\theta \in L^2(0,T)$ such that the solution of (H) satisfies

$$(w(\cdot,T), w_t(\cdot,T)) = (w^0, w^1).$$

Moreover,

$$\|\theta\|^2_{L^2(0,T)} \asymp \left\|w^0\right\|^2_{\frac{1-2\nu}{2}} + \left\|w^1\right\|^2_{\frac{-1-2\nu}{2}}.$$

- ✓ (Obs) is not true for $T < T_\alpha$. The case $T = T_\alpha$ remains open.
- ✓ The control θ is the unique control of minimal $L^2(0,T)$ norm.

Some elements of proof I

✓ We wright down the solutions of (H) in the form

$$v(x,t) = \sum_{n\in\mathbb{N}^*} v_n(t)\Phi_n(x), \quad \text{with} \quad v_n(t) = b_n e^{i\sqrt{\lambda_n}t} + b_{-n} e^{i\sqrt{\lambda_n}t}.$$

✓ The eigenfunctions in the neighborhood of 0 behave like

$$x^\alpha \Phi_{n,x}(x) \sim \frac{(1-\alpha)\kappa^{1/2} 2^{1/2-\nu} (j_{\nu,n})^\nu}{\Gamma(\nu+1)|J'_\nu(j_{\nu,n})|} \quad \text{as} \quad x \to 0^+.$$

✓ Behaviour of the zeros of Bessel's functions : $\inf_{k\neq l} |\sqrt{\lambda_k} - \sqrt{\lambda_l}| = \kappa(j_{\nu,2} - j_{\nu,1})$.

✓ We can apply Ingham inequality for $T > 2\pi/\kappa(j_{\nu,2} - j_{\nu,1})$

$$\sum_{k\geq 1} \frac{(j_{\nu,k})^\nu}{[J'_\nu(j_{\nu,k})]^2} \left(|v_k^0|^2 + \frac{|v_k^1|^2}{(\kappa j_{\nu,k})^2} \right) \asymp \int_0^T \left\{ x^\alpha \frac{\partial z}{\partial x}(x,t) \right\}^2_{|x=0} dt. \qquad \text{(Riesz)}$$

Some elements of proof II

✓ In fact Ingham inequality holds for $T > 2\pi D^+ = 4/(2-\alpha)$. Here, D^+ is the upper density of the sequence $(\kappa j_{\nu,k})_{k\in\mathbb{N}}$ defined by

$$D^+ := \lim_{r\to\infty} \frac{n^+(r)}{r}, \qquad (1)$$

where $n^+(r)$ denotes the largest number of terms of the sequence $(\kappa j_{\nu,k})_{k\in\mathbb{N}}$ contained in an interval of length r.

✓ Asymptotic behaviour of Bessel functions

$$|J'_\nu(j_{\nu,k})| \asymp \sqrt{\frac{2}{\pi j_{\nu,k}}}, \ \forall\ k \geq 1.$$

✓ Two sided inequality (hidden regularity)

$$\int_0^T \left\{ x^\alpha \frac{\partial z}{\partial x}(x,t) \right\}^2_{|x=0} \mathrm{d}t \asymp \sum_{k\geq 1} (\kappa j_{\nu,k})^{1+2\nu} \left| z_k^0 \right|^2 + (\kappa j_{\nu,k})^{-1+2\nu} \left| z_k^1 \right|^2. \qquad (\mathrm{Obs})$$

The case of the degenerate heat equation

For $\alpha \in [0,1)$ we consider the followng control problem

$$\begin{cases} y_t - (x^\alpha y_x)_x = 0 & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = \gamma(t),\ y(1,t) = 0 & t \in (0,T), \\ y(x,0) = y^0(x) & x \in (0,1). \end{cases} \qquad (\mathsf{P})$$

Théorème Let $\alpha \in [0,1)$, $y^0 \in \mathbf{H}_\alpha^{\frac{2\nu-1}{2}}$ and $T > 0$. Then, there exist constants C_0 et C_1 (independants of y^0 and T) such that the solution of (P) with $\gamma \equiv 0$ satisfies

$$\|y(T)\|^2_{\frac{2\nu-1}{2}} \le \frac{C_0}{T^2} \exp\left(\frac{C_1}{T}\right) \int_0^T \left\{ x^\alpha \frac{\partial z}{\partial x}(x,t) \right\}^2_{|x=0} \mathrm{d}t.$$

Théorème Let $\alpha \in (0,1)$. For all $y^0 \in L^2(0,1)$ and $T > 0$, there exists a control $\gamma \in L^2(0,T)$ such that the solution of (P) satisfies $y(\cdot,T) = 0$.

✓ The cost of controllability is similar to the one of the classical heat equation.

Some elements of proof

For any solution of (P) with $\gamma = 0$, $z_0 \in \mathbf{H}_\alpha^{\frac{1}{2}(2\nu-1)}$ and $L > T_\alpha/2$, we have

$$\left\| \int_0^T \partial_s k_T(0,t) z(\cdot,t) \mathrm{d}t \right\|^2_{\mathbf{H}_\alpha^{\frac{1}{2}(2\nu-1)}} \asymp \int_{-L}^{L} \left(\int_0^T k_T(s,t) \{x^\alpha z_x(x,t)\}_{|x=0} \mathrm{d}t \right)^2 \mathrm{d}s. \tag{Obs}$$

First, we estimate from above the term on the right hand side of (Obs). This is the easy part.
Next for the term on the right side of (Obs) we get

$$\int_{-L}^{L} \left(\int_0^T k_T(t,s) \{x^\alpha z_x(\cdot,t)\}_{|x=0} \mathrm{d}t \right)^2 \mathrm{d}s \le C \int_0^T \{x^{2\alpha} z_x^2(\cdot,t)\}_{|x=0} \mathrm{d}t,$$

where we have chosen $\delta \in (0,1)$ to be such that $(1+\delta)/\delta < \beta/L^2$. This is possible since $\beta > 2L^2$.

The case of operators in nondivergence form

For $\alpha \in [0,1)$ consider the differential expression defined by

$$\mathcal{A}y := -x^{\alpha} y''(x), \quad x \in (0,1).$$

✓ Now set the following weighted Sobolev space

$$L^2_{1/x^\alpha}(0,1) := \left\{ f \in L^2(0,1) \mid x^{-\alpha/2} f \in L^2(0,1) \right\}, \qquad \text{(WSob)}$$

$$H^1_{\alpha,0}(0,1) := L^2_{1/x^\alpha}(0,1) \cap H^1_0(0,1). \qquad \text{(WSob)}$$

$$D_{\max} := \left\{ f,\, f' \in AC_{\mathrm{loc}}(0,1) \mid f,\, x^{\alpha} f'' \in L^2_{1/x^\alpha}(0,1) \right\}$$

✓ Define the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset L^2_{1/x^\alpha}(0,1) \to L^2_{1/x^\alpha}(0,1)$

$$\begin{cases} D(\mathcal{A}) := \left\{ f \in H^1_{\alpha,0}(0,1) \,|\, x^{\alpha} f'' \in L^2_{1/x^\alpha}(0,1) \right\}, \\ \forall\, f \in D(\mathcal{A}),\ \mathcal{A}f := -x^{\alpha} f''. \end{cases} \qquad \text{(D)}$$

Conclusion and open problems

✓ In the regular case R the previous results extend to a certain range of $a \in \mathcal{C}^0([0,1]) \cap \mathcal{C}^1((0,1])$, $a > 0$ on $(0,1]$ and $a(0) = 0$ such that $1/a \in L^1(0,1)$.

✓ The singular case requires a special treatment. For instance for $1 < \alpha < 3/2$ the endpoint 0 is LC. there exist elements f, g in $D_{\max}$ such that $[f,g](0) \neq 0$.

✓ Interior degeneracy : designe a suitable self-adjoint extension $(\mathcal{A}, D(\mathcal{A}))$ of

$$\Big(-(|x|^\alpha y')', \quad C_0^\infty(-1,0) \cup (0,1) \Big) . \qquad \text{(Dint)}$$

✓ Linearizations of Prandt or Crocco-type equations for fluids, Kolmogorov. For instance Grushin type operators

$$\Big(-\partial_{xx} y - |x|^\alpha \partial_{zz} y + \frac{c_\alpha}{x^2} y, \quad C_0^\infty(-1,0) \cup (0,1) \Big) \qquad \text{(Gru)}$$

Merci de votre attention !