Minimal time of controllability for some parabolic systems

Manuel González-Burgos

In colaborationn with: F. Ammar-Khodja, A. Benabdallah and L. de Teresa

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GOAL

The general aim of this talk is to show a phenomenon which arise when we deal with the null controllability properties of **parabolic coupled** systems: **minimal time of controllability**:

1. **Boundary control:** The **condensation index** of the complex sequence of eigenvalues of the corresponding matrix elliptic operator.
2. **Distributed control:** The action and the geometric position of the support of the coupling term when this support does not intersect the control domain ω.

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1. Introduction. Statement of the problem

1 Introduction. Statement of the problem

Let us fix $T > 0$ and $\omega = (a, b) \subset (0, \pi)$. We consider the coupled parabolic systems:

(1) $$\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q := (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

(2) $$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

In (1) and (2), 1_ω is the characteristic function of the set ω, $y(x,t)$ is the state, $y_0 \in L^2(0,\pi;\mathbb{R}^2)$ (or $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$) is the initial datum and

- $D = \operatorname{diag}(d_1, d_2) \in \mathcal{L}(\mathbb{R}^2)$, with $d_i > 0$, and $A_0 \in \mathcal{L}(\mathbb{R}^2)$ constant matrices; $q \in L^\infty(Q)$; $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ constant vector of $\mathbb{R}^2$;
- $v \in L^2(0,T)$ and $u \in L^2(Q)$ are scalar control functions.

1 Introduction. Statement of the problem

Remark

In this talk we are interested in studying the controllability properties of systems (1) and (2). Boundary and distributed control problems.

IMPORTANT

We have systems of two coupled heat equations and we want to control these systems (two states) only acting on the second equation.

$$
(1) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}
$$

Existing results: $\boxed{d_1 = d_2}$: Approximate and null controllability.

- E. Fernández-Cara, M.G.-B., L. de Teresa, J. Funct. Anal. (2010): 2×2 systems, 1-d, general matrices of constant coefficients, necessary and sufficient conditions, boundary NC $\not\Leftrightarrow$ internal NC.
- F. Ammar-Khodja, A. Benabdallah, M.G.-B., L. de Teresa, J. Math. Pures Appl. (2011): $n \times n$ systems, 1-d, general matrices of constant coefficients, necessary and sufficient conditions, boundary NC $\not\Leftrightarrow$ internal NC.

1 Introduction. Statement of the problem

$$(1) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

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- L. Rosier , L. de Teresa, C. R. Math. Acad. Sci. Paris (2011), 2×2 systems, 1-d, cascade systems, sing conditions, sufficient conditions.
- F. Alabau-Boussouira, M. Léautaud, J. Math. Pures Appl. (2012): 2×2 systems, N-d, particular matrices depending on x, sing conditions, sufficient conditions, geometric control condition.
- F. Alabau-Boussouira, Math. Control Signals Systems (2014): 2×2 systems, N-d, cascade systems, sing conditions, sufficient conditions, geometric control condition.

$$(1) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

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- A. Benabdallah, F. Boyer, M.G.-B., G. Olive, *Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null-controllability in cylindrical domains*, (2014). Under review.

$$(2)\qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Existing results: $\boxed{\omega \cap \operatorname{Supp} q \neq \emptyset, \; a_{12} \neq 0}$: Approximate and null controllability.

- L. DE TERESA, Comm. PDE **25** (2000).
- F. AMMAR-KHODJA, A. BENABDALLAH, C. DUPAIX, I. KOSTIN, ESAIM:COCV (2005).
- M.G.-B., R. PÉREZ-GARCÍA, Asymptot. Anal. (2006).
- M.G.-B., L. DE TERESA, Port. Math. (2010).

Different diffusion coefficients, any space dimension.

$$\begin{cases} \partial_t y_1 - d_1 \partial_x^2 y_1 + a_{11} y_1 + \boxed{a_{12}} y_2 = 0 & \text{in } Q, \\ \partial_t y_2 - d_2 \partial_x^2 y_2 + a_{22} y_2 + a_{21} y_1 = u 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Existing results: a_{12} is a PD operator of order ≤ 2 with $\boxed{\omega \cap \mathrm{Supp}\, a_{12} \neq \emptyset}$ and a_{12} is "invertible": Approximate and null controllability.

- S. GUERRERO, SIAM J. Control Optim. **25** (2007).
- A. BENABDALLAH, M. CRISTOFOL, P. GAITAN, L. DE TERESA, Math. Control Relat. Fields (2014).
- K. MAUFFREY, J. Math. Pures Appl. (2013).

Different diffusion coefficients, any space dimension.

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Existing results: $\omega \cap \operatorname{Supp} q = \emptyset$ and $a_{12} \neq 0$ (sign conditions).

- O. Kavian, L. de Teresa, ESAIM:COCV (2010): Approximate controllability.
- L. Rosier, L. de Teresa, C. R. Math. Acad. Sci. Paris (2011): Null controllability.
- F. Alabau-Boussouira, M. Léautaud, J. Math. Pures Appl. (2012): 2×2 systems, N-d, particular matrices depending on x, sing conditions, sufficient conditions, geometric control condition. Null controllability.

1 Introduction. Statement of the problem

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Existing results: $\boxed{\omega \cap \mathrm{Supp}\, q = \emptyset}$ and $a_{12} \neq 0$

- F. ALABAU-BOUSSOUIRA, Math. Control Signals Systems (2014): 2×2 systems, N-d, cascade systems, sing conditions, sufficient conditions, geometric control condition. Null controllability.
- F. BOYER, G. OLIVE, Mathematical Control and Related Fields (2014). Approximate controllability, **no sign conditions.**
- B. DEHMAN, M. LÉAUTAUD, J. LE ROUSSEAU, Arch. Rational Mech. Anal. (2014). Null controllability.

Objective

We want to study the controllability properties of systems (1) and (2):

$$\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$\begin{cases} y_t - y_{xx} + q(x) A_0 y = Bu 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

in the one-dimensional case $N = 1$ and under the assumptions:

1. $D = \operatorname{diag}(d_1, d_2)$ and $\boxed{d_1 \neq d_2}$.
2. $q \in L^\infty(Q)$ (**no sign conditions**).

We will consider the “simple" case: $\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

2. Boundary controllability problem

(1) $$\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $v \in L^2(0,T)$: scalar control function.

Theorem (Fernández-Cara, M.G.-B., de Teresa, (2010))

Assume $d_1 = d_2 > 0$. *Then system* (1) *is null controllable at time* T *for any* $T > 0$.

2 Boundary controllability problem

$$(1) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $v \in L^2(0,T)$: scalar control function.

Theorem (Fernández-Cara, M.G.-B., de Teresa, (2010))

Assume $d_1 = d_2 > 0$. Then system (1) is null controllable at time T for any $T > 0$.

We will assume that $d_1 \neq d_2$ and, for instance, $d_1 = 1$, $d_2 = d \neq 1$.

GOAL

Given $T > 0$, does there exist $v \in L^2(0,T)$ s.t. $y(T) = 0$?

2 Boundary controllability problem

(1) $$\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Approximate controllability:

Theorem (Fernández-Cara, M.G.-B., de Teresa, (2010))

Assume $d \neq 1$. Then system (1) is approximately controllable at time $T > 0$ if and only if $\sqrt{d} \notin \mathbb{Q}$.

A simple problem??? No:

2 Boundary controllability problem

$$(1) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Approximate controllability:

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A simple problem??? No:

Theorem (Luca, de Teresa, (2012))

There exists $d > 0$ with $\sqrt{d} \notin \mathbb{Q}$ such that system (1) is not null controllable at any time $T > 0$.

2 Boundary controllability problem

$$(1) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

Assumption

In the sequel, $\boxed{D = \text{diag}\,(1, d)}$ with $\boxed{d \neq 1}$.

2 Boundary controllability problem

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Let φ be a solution of the adjoint problem:

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^*\varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 \in H_0^1(0,\pi)^2 & \text{in } (0,\pi). \end{cases}$$

If y is a solution of the direct problem, then

$$\langle y(T), \varphi_0 \rangle - \langle y_0, \varphi(0) \rangle = \int_0^T v(t) B^* D\varphi_x(0,t)\, dt$$

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If y is a solution of the direct problem, then

$$\langle y(T),\varphi_0\rangle - \langle y_0,\varphi(0)\rangle = \int_0^T v(t)B^*D\varphi_x(0,t)\,dt$$

Thus $\boxed{y(T)=0} \iff \exists v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(t)B^*D\varphi_x(0,t)\,dt = -\langle y_0,\varphi(0)\rangle\,, \quad \forall\varphi_0 \in H_0^1(0,\pi)^2}$$

Fattorini-Russell Method

Material at our disposal

- $\sigma(-D\partial_{xx}^2 + A_0^*) = \bigcup_{k\geq 1}\{k^2, dk^2\} := \bigcup_{k\geq 1}\{\lambda_{k,1}, \lambda_{k,2}\}$
- $V_{k,1}$ and $V_{k,2}$: eigenvectors of the matrix $(k^2 D + A_0^*)$ associated to the eigenvalues k^2, dk^2.
- $\Phi_{k,i} = V_{k,i}\sin kx$, $i = 1, 2$: eigenfunctions of $(-D\partial_{xx}^2 + A_0^*)$.
- $\{\Phi_{k,i}\}$ is a (Riesz) basis of $H_0^1(0,\pi)^2$. Let $\{\Psi_{k,i}\}$ be the associated biorthogonal family (for the duality $\langle\cdot,\cdot\rangle_{((H_0^1)^2,(H^{-1})^2)}$)

$$f \in H_0^1(0,\pi)^2 \Longleftrightarrow f = \sum_{k\geq 1, i=1,2} \langle f, \Psi_{k,i}\rangle \Phi_{k,i}$$

$$\|f\|^2_{(H_0^1)^2} \sim \sum_{k\geq 1, i=1,2} |\langle f, \Psi_{k,i}\rangle|^2$$

2 Boundary controllability problem

$$(1) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Objective: Existence of $v \in L^2(0,T)$ s.t.

$$\int_0^T v(t) B^* D\varphi_x(0,t)\, dt = -\langle y_0, \varphi(0)\rangle, \quad \forall \varphi_0 \in H_0^1(0,\pi)^2$$

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- Choosing $\varphi_0 = \Phi_{k,i}$, we have $\varphi(\cdot,t) = e^{-\lambda_{k,i}(T-t)} \Phi_{k,i}$ and

$$\varphi(x,0) = e^{-\lambda_{k,i} T} \Phi_{k,i}(x), \quad \varphi_x(0,t) = k e^{-\lambda_{k,i}(T-t)} V_{k,i}$$

- The identity connecting y and φ writes (**moment problem**)

$$k B^* D V_{k,i} \int_0^T v(T-t) e^{-\lambda_{k,i} t}\, dt = -e^{-\lambda_{k,i} T} \langle y_0, \Phi_{k,i} \rangle, \quad \forall (k,i)$$

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Approximate controllability: a necessary condition (I)

- $$\boxed{\boxed{kB^* DV_{k,i}} \int_0^T v(T-t) e^{-\lambda_{k,i} t}\, dt = -e^{-\lambda_{k,i} T} \left\langle y_0, \Phi_{k,i} \right\rangle, \quad \forall (k,i)}$$

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- A necessary condition: $B^*DV_{k,i} \neq 0$ for all $k \geq 1, i = 1,2$
- Recall $\boxed{d \neq 1}$,

$$B^* = (0,1), \quad V_{k,1} = \begin{pmatrix} 1 \\ \frac{1}{(d-1)k^2} \end{pmatrix}, \quad V_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \forall k \geq 1.$$

So, here $\boxed{B^*DV_{k,i} \neq 0, \quad \forall k \geq 1, i = 1,2}$

$$(1)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Approximate controllability: a necessary condition (II)

$$\lambda_{k,1} = \lambda_{j,2} = \lambda \Rightarrow \begin{cases} kB^* DV_{k,1} \int_0^T v(T-t) e^{-\lambda t}\, dt = -e^{-\lambda T} \langle y_0, \Phi_{k,1} \rangle \\ jB^* DV_{j,2} \int_0^T v(T-t) e^{-\lambda t\, dt} = -e^{-\lambda T} \langle y_0, \Phi_{j,2} \rangle \end{cases}$$

So it is necessary to have $\lambda_{k,1} \neq \lambda_{j,2}$. This leads to

$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \iff \boxed{\sqrt{d} \notin \mathbb{Q}}$$

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So it is necessary to have $\lambda_{k,1} \neq \lambda_{j,2}$. This leads to

$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \iff \boxed{\sqrt{d} \notin \mathbb{Q}}$$

In the sequel, we will assume $\sqrt{d} \notin \mathbb{Q}$, i.e., the eigenvalues of $-D\partial_{xx}^2 + A_0^*$ with Dirichlet boundary conditions are pairwise distinct.

2 Boundary controllability problem

$$(1) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$kB^* DV_{k,i} \int_0^T v(T-t) e^{-\lambda_{k,i} t}\, dt = -e^{-\lambda_{k,i} T} \left\langle y_0, \Phi_{k,i} \right\rangle, \quad \forall (k,i)$$

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$$kB^* DV_{k,i} \int_0^T v(T-t) e^{-\lambda_{k,i} t}\, dt = -e^{-\lambda_{k,i} T} \langle y_0, \Phi_{k,i} \rangle, \quad \forall (k,i)$$

Summarizing

Let $m_{k,i} = -\langle y_0, \Phi_{k,i} \rangle$, $b_{k,i} = kB^* DV_{k,i}$ (for any $\varepsilon > 0$, $\boxed{|m_{k,i}| \le C_\varepsilon e^{\varepsilon \lambda_{k,i}}}$ and $\boxed{|b_{k,i}| \ge C_\varepsilon e^{-\varepsilon \lambda_{k,i}}}$),

$$\exists ?\, v \in L^2(0,T) \, : \, \int_0^T v(T-t) e^{-\lambda_{k,i} t}\, dt = \frac{m_{k,i}}{b_{k,i}} e^{-\lambda_{k,i} T}, \quad \forall k \ge 1, \ i = 1,2$$

Let $\Lambda = \{\lambda_k\}_{k\geq 1} \subset (0,\infty)$ be a sequence with **pairwise distinct elements**:

$$\boxed{\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty}$$

Goal: Given $\{m_k\}_{k\geq 1}, \{b_k\}_{k\geq 1} \subset \mathbb{R}$ satisfying $\boxed{|m_k| \leq C_\varepsilon e^{\varepsilon\lambda_k}}$ and $\boxed{|b_k| \geq C_\varepsilon e^{-\varepsilon\lambda_k}}$, find $v \in L^2(0,T)$ s.t.

$$\int_0^T v(T-t)e^{-\lambda_k t}\,dt = \frac{m_k}{b_k}e^{-\lambda_k T}, \quad \forall k \geq 1.$$

Theorem

Under the previous assumptions, $\{e^{-\lambda_k t}\}_{k\geq 1} \subset L^2(0,T)$ *admits a* ***biorthogonal family*** $\{q_k\}_{k\geq 1}$ *in* $L^2(0,T)$*, i.e.:*

$$\int_0^T e^{-\lambda_k t} q_l(t)\,dt = \delta_{kl}, \quad \forall k,l \geq 1$$

A formal solution to

$$\int_0^T v(T-t)e^{-\lambda_k t}\,dt = \frac{m_k}{b_k}e^{-\lambda_k T}, \quad \forall k \geq 1,$$

is v given by: $$v(T-t) = \sum_{k\geq 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t),$$

A formal solution to

$$\boxed{\int_0^T v(T-t)e^{-\lambda_k t}\,dt = \frac{m_k}{b_k}e^{-\lambda_k T}, \quad \forall k \geq 1,}$$

is v given by: $\boxed{v(T-t) = \sum_{k\geq 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t)}$,

Question: $v \in L^2(0,T)$?, i.e., is the series $\boxed{\sum_{k\geq 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t)}$ convergent in $L^2(0,T)$?

But this question itself amounts to:

$$\|q_k\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$$

Theorem

Assume

$$\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty.$$

Then, for any $\varepsilon > 0$ *one has*

$$C_{1,\varepsilon} \frac{e^{-\varepsilon \lambda_k}}{|E'(\lambda_k)|} \leq \|q_k\|_{L^2(0,T)} \leq C_{2,\varepsilon} \frac{e^{\varepsilon \lambda_k}}{|E'(\lambda_k)|}, \quad \forall k \geq 1,$$

where $E(z)$ *is the interpolating function:*

$$E(z) = \prod_{k=1}^{\infty} (1 - \frac{z^2}{\lambda_k^2}), \qquad \boxed{E'(\lambda_k) = -\frac{2}{\lambda_k} \prod_{j\neq k}^{\infty} \left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right)}$$

Definition

The **index of condensation** of $\Lambda = \{\lambda_k\}_{k \geq 1} \subset \mathbb{C}$ is:

$$c(\Lambda) = \limsup_{k \to \infty} \frac{-\ln |E'(\lambda_k)|}{\Re(\lambda_k)} \in [0, +\infty] .$$

Corollary

For any $\varepsilon > 0$ one has

$$\|q_k\|_{L^2(0,T;)} \leq C_\varepsilon e^{(c(\Lambda)+\varepsilon)\lambda_k}, \quad \forall k \geq 1.$$

Recall that we had m_k s.t. $|m_k| \leq C_\varepsilon e^{\varepsilon \lambda_k}$, $|b_k| \geq C_\varepsilon e^{-\varepsilon \lambda_k}$, for any $\varepsilon > 0$, and we wanted to solve: $v \in L^2(0,T)$ and

$$\boxed{\int_0^T v(T-t) e^{-\lambda_k t}\, dt = \frac{m_k}{b_k} e^{-\lambda_k T}, \quad \forall k},$$

We took $v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t)$.

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We took $v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t)$.

From the previous result: Given $\varepsilon > 0$:

$$\left| \frac{m_k}{b_k} \right| e^{-\lambda_k T} \, \|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{-\lambda_{k,i}(T - c(\Lambda) - \varepsilon)}$$

Then

$$T > c(\Lambda) \Longrightarrow v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t) \in L^2(0,T).$$

$$
(1) \qquad \begin{cases} y_t - D y_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

In our case,

$$
\Lambda_d := \{\lambda_k\}_{k\geq 1} = \left\{ j^2, dj^2 \right\}_{j\geq 1}.
$$

Then

If $T > c(\Lambda_d)$, system (1) is null controllable at time T, where $c(\Lambda_d)$ is the **index of condensation** of the sequence Λ_d.

$$(1) \qquad \begin{cases} y_t - D y_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

- The **index of condensation** of a sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ is a real number $c(\Lambda) \in [0,+\infty]$ associated with this sequence and which "measures" the condensation at infinity.

$$c(\Lambda) = \limsup_{k\to\infty} \frac{-\ln|E'(\lambda_k)|}{\Re(\lambda_k)} \in [0,\infty], \quad \boxed{E'(\lambda_k) = \frac{-2}{\lambda_k}\prod_{j\neq k}^{\infty}\left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right)}.$$

- This notion has been :
  - introduced by V.I. Bernstein in 1933: Leçons sur les progrès récents de la théorie des séries de Dirichlet for real sequences,
  - extended by J. R. Shackell in 1967 for complex sequences.

1. **Gap property**: $\exists \rho > 0 \,:\, |\lambda_k - \lambda_l| \geq \rho\, |k - l| \Rightarrow \boxed{c(\Lambda) = 0}$.
   In particular: for the scalar Dirichlet-Laplacien operator: $\lambda_k = k^2$, $|\lambda_k - \lambda_l| = |k^2 - l^2| \geq |k - l|$. So
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2. $\alpha > 1$, $\beta > 0$ and $\Lambda = \{\lambda_k\}_{k \geq 1}$ with $\lambda_{2k} = k^\alpha$, $\lambda_{2k+1} = k^\alpha + e^{-k^\beta}$
   $$c(\Lambda) = \begin{cases} 0 & \beta < \alpha \\ 1 & \beta = \alpha \\ +\infty & \beta > \alpha \end{cases} \qquad \text{(Note that } \boxed{\liminf |\lambda_{k+1} - \lambda_k| = 0}\text{)}$$

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3. $\Lambda = \{\lambda_k\}_{k\geq 1}$ with
   $$\lambda_{k^2+n} = k^2 + ne^{-k^2}, \quad n \in \{0, \cdots, 2k\}, \quad k \geq 1$$
   $$\boxed{c(\Lambda) = +\infty}$$

$$(1)\qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

$$\Lambda_d = \{k^2, dk^2\}_{k\geq 1}, \quad \sqrt{d} \notin \mathbb{Q}.$$

We have proved:

Theorem

There exists $T_0 = c(\Lambda_d) \in [0, +\infty]$ *such that if* $\boxed{T > T_0}$ *then system* (1) *is null controllable at time* T

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Theorem

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$\boxed{T > c(\Lambda_d)}$ is a sufficient condition for the null controllability of system (1) at time T. But,

what happens if $\boxed{T < c(\Lambda_d)}$?

$$(1)\qquad \left\{\begin{array}{ll} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{array}\right.$$

The null controllability property at time T of system (1) is equivalent to the **observability inequality**:

$$\|\varphi(\cdot,0)\|^2_{(H_0^1)^2} \le C_T \int_0^T \left|B^* D \partial_x \varphi(0,t)\right|^2 \, dt,$$

for the solutions to **the adjoint problem**

$$\left\{\begin{array}{ll} -\varphi_t - D\varphi_{xx} + A_0^* \varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{array}\right.$$

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^*\varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}$$

- $\sigma(-D\partial_{xx}^2 + A_0^*) = \bigcup_{k\geq 1}\{k^2, dk^2\} := \bigcup_{k\geq 1}\{\lambda_{k,1}, \lambda_{k,2}\}$
- $V_{k,1}$ and $V_{k,2}$: eigenvectors of the matrix $(k^2 D + A_0^*)$ associated to the eigenvalues k^2, dk^2.
- $\Phi_{k,i} = V_{k,i} \sin kx$, $i = 1,2$: eigenfunctions of $(-D\partial_{xx}^2 + A_0^*)$.
- $\{\Phi_{k,i}\}$ is a (Riesz) basis of $H_0^1(0,\pi)^2$. Let $\{\Psi_{k,i}\}$ be the associated biorthogonal family (for the duality $\langle\cdot,\cdot\rangle_{((H_0^1)^2,(H^{-1})^2)}$)

$$f \in H_0^1(0,\pi)^2 \Longleftrightarrow f = \sum_{k\geq 1, i=1,2} \langle f, \Psi_{k,i}\rangle \Phi_{k,i}$$

$$\|f\|^2_{(H_0^1)^2} = \sum_{k\geq 1, i=1,2} |\langle f, \Psi_{k,i}\rangle|^2$$

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^*\varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}$$

Thus, the **observability inequality** for the adjoint system writes

$$\sum_{n,i} e^{-2\lambda_{n,i}T} |a_{n,i}|^2 \leq C_T \int_0^T \left| \sum_{n,i} nB^* D V_{n,i} e^{-\lambda_{n,i}t} a_{n,i} \right|^2 dt,$$

$\forall \{a_{n,i}\}_{n,i} \in \ell^2$.

$$\sum_{n,i} e^{-2\lambda_{n,i}T}|a_{n,i}|^2 \le C_T \int_0^T \left|\sum_{n,i} nB^* D V_{n,i} e^{-\lambda_{n,i}t} a_{n,i}\right|^2 dt,$$

Assume $T \in (0, c(\Lambda_d))$.

By contradiction: Assume the **observability inequality** holds for $C_T > 0$

Construction of a suitable sequence of initial data

The idea is to construct sequences $\{a_{n,i}^{(k)}\}_{n,i} \in \ell^2$ such that

$$\int_0^T \left|\sum_{n,i} nB^* D V_{n,i} e^{-\lambda_{n,i}t} a_{n,i}^{(k)}\right|^2 \to 0, \quad \sum_{n,i} e^{-2\lambda_{n,i}T}|a_{n,i}^{(k)}|^2 \ge \delta > 0.$$

Argument: Use the overconvergence of Dirichlet series

Theorem

Suppose that the sequence $\Lambda = \{\lambda_n\}_{n\geq 1}$ has index of condensation $c(\Lambda)$. We can choose a sequence of finite sets $N_k \subset \mathbb{N}$, a sequence $\{\alpha_n\}_{n\geq 1} \subset \mathbb{C}$, such that there exists $R \geq 0$ such that

1. *the series $\sum_{n\geq 1} \alpha_n e^{-\lambda_n z}$ converges in the region $\Re z > R$*
2. *the series $\sum_{n\geq 1} \alpha_n e^{-\lambda_n z}$ diverges in the region $\Re z < R$*
3. *the series $\sum_{k\geq 1}(\sum_{n\in N_k} \alpha_n e^{-\lambda_n z})$ converges in the region $\Re z > R - c(\Lambda)$*

- One can construct $\{\alpha_n\}_{n\geq 1}$ such that $R = c(\Lambda)$.
- The construction of the sequence $\{\alpha_n\}_{n\geq 1}$ is explicit.

- $\Lambda_d = \{\lambda_n\}_{n\geq 1} = \{k^2, dk^2\}_{k\geq 1}$. We construct $\{a_n^{(k)}\}_{n\geq 1} \in \ell^2$:

$$a_n^{(k)} = \begin{cases} \frac{\alpha_n}{b_n} & n \in N_k \\ 0 & n \notin N_k \end{cases}$$

$$b_n = n\left|B^* D V_n\right|$$

- $\{a_n^{(k)}\}_{n\geq 1} \in \ell^2$ (recall that the sets N_k are finite).
- The **observability inequality** is

$$\boxed{\sum_{n\in N_k} e^{-2\lambda_n T}|a_n^{(k)}|^2 \leq C_T \int_0^T \left|\sum_{n\in N_k} e^{-\lambda_n t}\alpha_n\right|^2 dt,}$$

$$\sigma_1^{(k)} := \sum_{n\in N_k} e^{-2\lambda_n T}|a_n^{(k)}|^2 \leq C_T \int_0^T \left| \sum_{n\in N_k} e^{-\lambda_n t}\alpha_n \right|^2 dt := \sigma_2^{(k)},$$

- The convergence of the series $\sum_{k\geq 1}(\sum_{n\in N_k} \alpha_n e^{-\lambda_n t})$ for all $t > 0$ (recall that $R = c(\Lambda_d)$ and then $R - c(\Lambda_d) = 0$) implies:

$$\lim_{k\to+\infty} \sum_{n\in N_k} \alpha_n e^{-\lambda_n t} = 0, \quad \forall t > 0$$

$$\sigma_1^{(k)} := \sum_{n\in N_k} e^{-2\lambda_n T}|a_n^{(k)}|^2 \leq C_T \int_0^T \left| \sum_{n\in N_k} e^{-\lambda_n t}\alpha_n \right|^2 dt := \sigma_2^{(k)},$$

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$$\lim_{k\to+\infty} \sum_{n\in N_k} \alpha_n e^{-\lambda_n t} = 0, \quad \forall t > 0$$

- Moreover, one can prove there exist $C_1, C_2 > 0$ such that

$$\left| \sum_{n\in N_k} \alpha_n e^{-\lambda_n t} \right| \leq C_1 e^{-C_2 t}.$$

- Thus, from Lebesgue's dominated convergence theorem, we obtain $\sigma_2^{(k)} \to 0$.

$$\sigma_1^{(k)} := \sum_{n\in N_k} e^{-2\lambda_n T}|a_n^{(k)}|^2 \le C_T \int_0^T \left| \sum_{n\in N_k} e^{-\lambda_n t}\alpha_n \right|^2 dt := \sigma_2^{(k)},$$

- By construction the sequence $\{\alpha_n\}_{n\ge 1}$ satisfies that for all $k \ge 1$ there exists $n_k \in N_k$ such that

$$\left|a_{n_k}^{(k)}\right| = \left|\frac{\alpha_{n_k}}{b_{n_k}}\right| \ge C_\varepsilon e^{\Re(\lambda_{n_k})(c(\Lambda_d)-\varepsilon)}$$

- One gets:

$$\sigma_1^{(k)} \ge e^{-2\lambda_{n_k} T}\left|a_{n_k}^{(k)}\right|^2 \ge C_\varepsilon e^{2\Re(\lambda_{n_k})(c(\Lambda_d)-T-\varepsilon)} \underset{T<c(\Lambda_d)}{\longrightarrow} +\infty .$$

- So, one has proved

$$\sigma_1^{(k)} \to +\infty, \quad \sigma_2^{(k)} \to 0$$

$$(1) \quad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

The controllability result

1. $\forall T > 0$: **Approximate controllability** if and only if $\sqrt{d} \notin \mathbb{Q}$
2. Assume $\sqrt{d} \notin \mathbb{Q}$, $\exists T_0 = c(\Lambda_d) \in [0, +\infty]$ such that
   1. the system is null controllable at time T if $\boxed{T > T_0}$
   2. Even if $\sqrt{d} \notin \mathbb{Q}$, if $\boxed{T < T_0}$ the system is **not null controllable** at time T!

(1) $$\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

In fact, the good minimal time is

$$T_0 = \limsup_{k\to\infty} \frac{-\left(\ln|b_k| + \ln|E'(\lambda_k)|\right)}{\Re(\lambda_k)} \in [0,\infty]$$

2 Boundary controllability problem

$$(1)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$T_0 > 0$?

Is it possible to have a minimal time of control > 0? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

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Is it possible to have a minimal time of control > 0? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k \geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

Theorem

For any $\tau \in [0, +\infty]$, there exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = \tau$.

Remark

- There exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = +\infty$ (LUCA, DE TERESA).
- $c(\Lambda_d) = 0$ for almost $d \in (0, \infty)$ such that $\sqrt{d} \notin \mathbb{Q}$.
- For any $\tau \in [0, +\infty]$, the set $\{d \in (0, \infty) : c(\Lambda_d) = \tau\}$ is dense in $(0, +\infty)$.

2 Boundary controllability problem

F. AMMAR KHODJA, A. BENABDALLAH, M.G.-B., L. DE TERESA, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, under review (2014?).

http://personal.us.es/manoloburgos

Let us consider the corresponding distributed control problem

$$(3)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = Bv1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $v \in L^2(Q)$ is the control.

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where $v \in L^2(Q)$ is the control. One has:

Theorem (Distributed control)

System (3) *is* ***null controllable*** *at time T* ***if and only if***

$$(4) \qquad \det\left[B\,,\,(k^2 D + A_0)B\right] \neq 0, \quad \forall k \geq 1.$$

F. Ammar Khodja, A. Benabdallah, C. Dupaix, M.G.-B., J. Evol. Eq. (2009).

In our case, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $D = \mathrm{diag}\,(1, d)$. Thus,

$$(3) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = Bv1_\omega & \text{in } Q, \\ y(0,\cdot) = y(\pi,\cdot) = 0 \text{ on } (0,T), \quad y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

is null controllable at time T, for any $\boxed{T > 0}$ and any open set $\omega \subset (0,\pi)$.

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The minimal time for parabolic systems, is it typically a phenomenon of **boundary controllability problems**?? **NO!!**.

3. Distributed controllability problem

3 Distributed controllability problem

$$(2) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

where $q \in L^\infty(Q)$,

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$\omega = (a,b) \subset (0,\pi)$ and $v \in L^2(Q)$ is a scalar control function.

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$\omega = (a,b) \subset (0,\pi)$ and $v \in L^2(Q)$ is a scalar control function.

No sign conditions on q.

$$\omega \cap \mathrm{Supp}\, q = \emptyset$$

3 Distributed controllability problem

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Theorem (Ammar Khodja, Benabdallah, G-B, de Teresa (2011))

Assume $I_k(q) \neq 0$ *for any* $k \geq 1$, *where*

$$(5) \qquad I_k(q) := \int_0^\pi q(x) \sin^2(kx)\, dx,$$

and

$$\int_0^\pi q(x)\, dx \neq 0.$$

Then, for any $T > 0$, *system* (2) *is* ***null controllable*** *at time* T.

3 Distributed controllability problem

(2) $$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Null controllability properties of system (2) when

$$\int_0^\pi q(x)\,dx = 0?$$

In order to simplify the problem, we will assume the **geometrical assumption**:

Assumption (A1)

The function q satisfies $\mathrm{Supp}\, q \subset [0,a]$ or $\mathrm{Supp}\, q \subset [b,\pi]$ ($\omega = (a,b)$).

3 Distributed controllability problem

(2)
$$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Proposition (Boyer and Olive (2014))

Under the geometrical assumption (A1), system (2) is ***approximately controllable*** *at time $T > 0$ if and only if*

$$I_k(q) \neq 0, \quad \forall k \geq 1.$$

$$I_k(q) = \frac{1}{2}\int_0^\pi q(x)\,dx - \frac{1}{2}\int_0^\pi q(x)\cos(2kx)\,dx,$$

$$L^* := -\frac{d^2}{dx^2} + q(x)A_0^* : L^2(0,\pi)^2 \longrightarrow L^2(0,\pi)^2$$

domain $D(L^*) = H^2(0,\pi)^2 \cap H_0^1(0,\pi)^2$.

Lemma

The spectrum of L^* is given by $\sigma(L^*) = \{\lambda_k := k^2 : k \geq 1\}$. Moreover, λ_k is simple if and only if $I_k(q) \neq 0$, where

(5) $$I_k(q) := \int_0^\pi q(x) \sin^2(kx)\, dx.$$

Finally, if $I_k(q) = 0$, the eigenvalue λ_k of L^* is double.

3 Distributed controllability problem

$$L^* := -\frac{d^2}{dx^2} + q(x)A_0^* : L^2(0,\pi)^2 \longrightarrow L^2(0,\pi)^2$$

Proposition ($I_k(q) \neq 0$)

If

$$\Phi^*_{k,1} = \begin{pmatrix} \phi_k \\ I_k(q)\psi_k \end{pmatrix}, \quad \Phi^*_{k,2} = \begin{pmatrix} 0 \\ I_k(q)\phi_k \end{pmatrix},$$

where $\phi_k(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \sin kx$ *and* ψ_k *is the* ***unique solution*** *of*

$$(6) \qquad \begin{cases} -\psi_{xx} = \lambda_k \psi + \left[1 - I_k(q)^{-1} q(x)\right] \phi_k \text{ in } (0,\pi), \\ \psi(0) = 0, \quad \psi(\pi) = 0, \\ \int_0^\pi \psi(x)\phi_k(x)\, dx = 0, \end{cases}$$

then,

$$(L^* - \lambda_k I_d)\, \Phi^*_{k,1} = \Phi^*_{k,2}, \quad (L^* - \lambda_k I_d)\, \Phi^*_{k,2} = 0.$$

3 Distributed controllability problem

$$(2) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bf(x)v(t) & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Idea:

We will work with controls $u(x,t) = f(x)v(t)$ with $v \in L^2(0,T)$ and $f \in L^2(0,\pi)$ (appropriate) satisfies $\mathrm{Supp}\, f \subset \omega$.

Objective

Apply Fattorini-Russell method: **moment problem**

Basis of $L^2(0,\pi)^2$: $\mathcal{B} := \left\{ \Phi^*_{k,1}, \Phi^*_{k,2} \right\}_{k \geq 1}$.

3 Distributed controllability problem

$$(2)\qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bf(x)v(t) & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

The moment problem

Find $v \in L^2(0,T)$ s.t.

$$\begin{cases} \displaystyle\int_0^T v(T-t)\boxed{e^{-k^2 t}}\,dt = \frac{m_{k,1}}{b_{k,1}} e^{-k^2 T}, \quad \forall k \geq 1, \\ \displaystyle\int_0^T v(T-t)\boxed{te^{-k^2 t}}\,dt = \frac{m_{k,2}}{I_k(q) b_{k,2}} e^{-k^2 T}, \quad \forall k \geq 1, \end{cases}$$

where $\boxed{|m_{k,i}| \leq C_\varepsilon e^{\varepsilon \lambda_k}}$ and $\boxed{|b_{k,i}| \geq C_\varepsilon e^{-\varepsilon \lambda_k}}$ $(i = 1, 2)$.

$$(2)\qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Theorem

Assume $I_k(q) \neq 0$ for all $k \geq 1$ and let:

$$T_0(q) = T_0 := \limsup \frac{-\ln |I_k(q)|}{k^2} \in [0,+\infty].$$

Then, if $T > T_0$, system (2) is null-controllable at time T.

What happens if $\boxed{T < T_0(q)}$?

$$(2) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

As before, the null controllability property for system (2) is equivalent to the **observability inequality**:

$$\|\varphi(\cdot,0)\|^2_{(L^2)^2} \leq C_T \int_0^T \int_\omega |\varphi_2(x,t)|^2 \, dx\, dt,$$

for the solutions to **the adjoint problem**

$$\begin{cases} -\varphi_t - \varphi_{xx} + q(x)A_0^* \varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}$$

$$\|\varphi(\cdot,0)\|^2_{(L^2)^2} \le C_T \int_0^T \int_\omega |\varphi_2(x,t)|^2 \, dx\, dt,$$

Again, we prove that the inequality does not hold.

Important:

Behavior of ψ_k in ω:

$$\psi_k(x) = \tau_k \sin(kx) + g_k(x), \quad \forall x \in \omega;$$

g_k is **bounded** in ω and $I_k(q)\tau_k \to 0$.

$$\varphi_0 = \Phi^*_{k,1} - \tau_k \Phi^*_{k,2} = \begin{pmatrix} \phi_k \\ I_k(q)\psi_k \end{pmatrix} - \tau_k \begin{pmatrix} 0 \\ I_k(q)\phi_k \end{pmatrix}$$

$$\frac{1}{2}e^{-2k^2T} \le \|\varphi(\cdot,0)\|^2_{(L^2)^2} \le C_T \int_0^T \int_\omega |\varphi_2(x,t)|^2 \, dx \, dt \le C_T I_k(q)^2$$

In particular

$$\boxed{1 \le C_T e^{2k^2T} I_k(q)^2 \equiv C_T e^{-2k^2\left(\frac{-\ln |I_k(q)|}{k^2} - T\right)}, \quad \forall k \ge 1.}$$

Recall

$$0 < T < T_0(q) = \limsup \frac{-\ln |I_k(q)|}{k^2}.$$

Choosing a subsequence, we get a contradiction.

3 Distributed controllability problem

$$\frac{1}{2}e^{-2k^2T} \leq \|\varphi(\cdot,0)\|^2_{(L^2)^2} \leq C_T \int_0^T \int_\omega |\varphi_2(x,t)|^2 \, dx\, dt \leq C_T I_k(q)^2$$

In particular

$$\boxed{1 \leq C_T e^{2k^2T} I_k(q)^2 \equiv C_T e^{-2k^2\left(\frac{-\ln|I_k(q)|}{k^2} - T\right)}, \quad \forall k \geq 1.}$$

Recall

$$0 < T < T_0(q) = \limsup \frac{-\ln|I_k(q)|}{k^2}.$$

Choosing a subsequence, we get a contradiction. Then system

(2) $$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

is not null controllable at time T.

3 Distributed controllability problem

$$(2) \quad \begin{cases} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Theorem

Assume $I_k(q) \neq 0$ for all $k \geq 1$ and let:

$$T_0(q) = T_0 := \limsup \frac{-\ln |I_k(q)|}{k^2} \in [0, +\infty]$$

Then,

1. *If $T > T_0$, then system (2) is null-controllable at time T.*
2. *If $\operatorname{Supp} q \subset [0, a]$ or $\operatorname{Supp} q \subset [b, \pi]$, for any $T < T_0$, the system is not null-controllable at time T.*

3 Distributed controllability problem

$$(2) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Question

Does there exist $q \in L^\infty(Q)$ such that $T_0(q) > 0$?

$$(2)\qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Question

Does there exist $q \in L^\infty(Q)$ such that $T_0(q) > 0$?

Theorem

For any $\tau \in [0,+\infty]$, there exists $q \in L^\infty(0,\pi)$ such that $T_0(q) = \tau$.

Note that if $\displaystyle\int_0^\pi q(x)\,dx \neq 0$ then $T_0(q) = 0$. In particular, the previous result recovers the results on null controllability of system (2) when a sign condition is imposed on q.

$$(2)\qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

F. Boyer, G. Olive, Mathematical Control and Related Fields (2014).
Assume $\omega \cap \operatorname{Supp} q = \emptyset$. Approximate controllability:

- A **necessary** and **sufficient condition** for the approximate controllability of system (2) at time T.
- System (2) is approximately controllable at a given time $T_0 > 0$ **if and only if** it is approximately controllable at any time $T > 0$.
- The **necessary** and **sufficient condition** strongly **depends on the relative position** of ω with respect to $\operatorname{Supp} q$.

(2)
$$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

- The null controllability result is valid (with a different **minimal time**) without the geometrical

Assumption (A1)

The function q satisfies $\mathrm{Supp}\, q \subset [0,a]$ or $\mathrm{Supp}\, q \subset [b,\pi]$ ($\omega = (a,b)$).

Comments

- This minimal time also arises in other parabolic problems (degenerated problems):
  BEAUCHARD, CANNARSA, GUGLIELMI, Null controllability of Grushin-type operators in dimension two. J. Eur. Math. Soc. (JEMS) (2014).

- The minimal time for parabolic systems imply negative controllability results for cascade hyperbolic systems when the coupling coefficient does not have constant sign (Alabau-Boussouira-Léautaud, Rosier-de Teresa, Dehman et al.)

F. AMMAR KHODJA, A. BENABDALLAH, M.G.-B., L. DE TERESA, *Minimal time of controllability of two parabolic equations with disjoint control and coupling domains*, C. R. Math. Acad. Sci. Paris, (2014).

http://personal.us.es/manoloburgos

Comments

Scalar case versus systems

	SCALAR CASE	SYSTEMS
minimal time of controls	No	Yes
approximate ⇔ null controllability	Yes	No
boundary ⇔ distributed control	Yes	No
geometrical conditions	No	Yes

Thank you for your attention!!