Time optimal control for heat equations Progresses and comments

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Formulation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^2 boundary $\partial\Omega$. Let $\omega \subset \Omega$ be an open subset with its characteristic function χ_ω. Consider

$$\begin{cases} y_t - \Delta y + a(x,t)y = \chi_\omega u & \text{in } \Omega \times \mathbb{R}^+, \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \qquad (1.1)$$

where $a \in L^\infty(\Omega \times \mathbb{R}^+)$, $y_0 \in L^2(\Omega) \setminus \{0\}$ and u is taken from the constraint set of controls:

$$\mathcal{U}^M \triangleq \{u : \mathbb{R}^+ \to L^2(\Omega) \text{ measurable } \mid \|u(t)\|_{L^2(\Omega)} \leq M \;\; a.e.\ t > 0\},$$

with $M > 0$. Write $y(\cdot\,; u)$ for the solution of (1.1).

Formulation

The time optimal control problem reads:

$$(TP)^M \qquad T(M) \triangleq \inf_{u\in\mathcal{U}^M} \{t > 0 : \; y(t;u) = 0\}.$$

$T(M)$ is the optimal time, $u^* \in \mathcal{U}^M$ is a time optimal control if $y(T(M);u^*) = 0$, and $u \in \mathcal{U}^M$ is an admissible control if $y(t;u) = 0$ for some $t > 0$.
In this problem, 0 is the target, while $\mathcal{U}^M$ is the constraint set of controls. In more general cases, the target set can be assumed to be a convex set in the state space; while the constraint set of controls can be assumed to be

$$\{u : \mathbb{R}^+ \to L^2(\Omega) \text{ measurable } \mid u(t) \in U \text{ for a.e. } t \in \mathbb{R}^+\},$$

with U a bounded and closed subset in the control space.
We focus ourself on the case where the target is 0.

Existence

We now introduce the existence of time optimal controls for $(TP)^M$. For this purpose, we define, for each $T > 0$, the following norm optimal control problem:

$$(NP)^T \qquad N(T) \triangleq \inf \left\{ \|u\|_{L^\infty(0,T;L^2(\Omega))} : \ y(T;u) = 0 \right\}.$$

By the null-controllability for heat equations, $(NP)^T$ has solutions. Besides, $\lim_{T\to\infty} N(T)$ exists. Write

$$\widehat{N} \triangleq \lim_{T\to\infty} N(T).$$

Existence

Theorem 1.1 (G.Wang, Y.Xu, Y. Zhang 2014)

Given $y_0 \in L^2(\Omega) \setminus \{0\}$, $(TP)^M$ has time optimal controls if and only if $M > \widehat{N}$.

The proof of this theorem is based on the bang-bang property of $(NP)^T$, i.e., any optimal control v^* to $(NP)^T$ verifies that

$$\|v^*(t)\|_{L^2(\Omega)} = N(T) \quad \text{for a.e. } t \in (0, T).$$

This bang-bang property was proved by K. D. Phung and G. Wang (2013).

Motivation

• In the state space $L^2(\Omega)$, the set $\{y(T(M);u) : u \in \mathcal{U}^M\}$ has no interior point. We do not know how to separate this set from the target set $\{0\}$ by a hyperplane in $L^2(\Omega)$. Thus, we do not know how to get the maximum principle for $(TP)^M$ by the way used in the case of O.D.E..

• The maximum principle + the unique continuation can imply the bang-bang property (B-B-P for short): any time optimal control u^* for $(TP)^M$ verifies $\|u^*(t)\|_{L^2(\Omega)} = M$ for a.e. $t \in (0, T(M))$.

• It is natural to ask if the B-B-P holds for $(TP)^M$.

The advantages from the B-B-P

(i) The uniqueness of time optimal control for $(TP)^M$.

(ii) $(TP)^M \Longleftrightarrow (NP)^{T(M)}$, i.e., they share the same optimal controls.

(iii) From the above equivalence, we can get a sufficient and necessary condition for (t^*, u^*) being the optimal time and optimal control pair to $(TP)^M$.

To state the aforementioned sufficient and necessary condition, we introduce the functional $J^T : L^2(\Omega) \to \mathbb{R}$ by setting

$$J^T(z) = \frac{1}{2}\Big(\int_0^T \|\chi_\omega \varphi(t;z)\|_{L^2(\Omega)}\, dt\Big)^2 + \langle \varphi(0;z), y_0\rangle_{L^2(\Omega),L^2(\Omega)},$$

for each $z \in L^2(\Omega)$, where $\varphi(\cdot\,;z)$ solves

$$\begin{cases} \varphi_t + \Delta\varphi - a\varphi = 0 & \text{in } \ \Omega \times (0,T), \\ \varphi = 0 & \text{on } \ \partial\Omega \times (0,T), \\ \varphi(T) = z & \text{in } \ \Omega. \end{cases} \tag{2.1}$$

We will extend $J^T(\cdot)$ over a bigger space such that it has a minimizer. (Since $J^T(\cdot)$ is not coercive in $L^2(\Omega)$, we are not able to show the existence of minimizers of J^T over $L^2(\Omega)$.) For this purpose, we define a norm over $L^2(\Omega)$ by setting

$$\|z\|_{X_T} \triangleq \int_0^T \|\chi_\omega \varphi(t;z)\|_{L^2(\Omega)}\, dt, \quad z \in L^2(\Omega).$$

Set

$$X_T = \overline{L^2(\Omega)}^{\|\cdot\|_{X_T}}. \tag{2.2}$$

Lemma 2.1

[G.Wang, E.Zuazua,, 2012] When $a = 0$, $J^T(\cdot)$ *has a unique minimizer* $\hat{z}$ *in* X_T. *The optimal control* $\hat{u}$ *to* $(NP)^T$ *reads:*

$$\hat{u}(t) = \Big(\int_0^T \|\chi_\omega \varphi(t)\|_{L^2(\Omega)}\, dt\Big) \frac{\chi_\omega \varphi(t)}{\|\chi_\omega \varphi(t)\|_{L^2(\Omega)}}, \text{ a.e. } t \in (0,T),$$

where φ *is the solution to Equ. (2.1) with the initial datum* $\hat{z}$.

This can be easily extended to the case where $a = a(x)$.

With the aid of Lemma 2.1 and the equivalence of $(TP)^M$ and $(NP)^{T(M)}$, we have

Theorem 2.2

[G.Wang, E.Zuazua, 2012] When $a = 0$, $t^* > 0$ is the optimal time for $(TP)^M$, if and only if

$$M = \int_0^{t^*} \|\widehat{\varphi}(t)\|_{L^2(\omega)}\, dt.$$

Here $\widehat{\varphi}(\cdot)$ solves the adjoint equation over $(0, t^*)$ with the initial condition $\widehat{\varphi}(t^*) = \widehat{\varphi}_{t^*}$, where $\widehat{\varphi}_{t^*}$ is the minimizer of J^{t^*}. Furthermore, the time optimal control u^* is characterized by

$$u^*(t) = M \frac{\chi_\omega \widehat{\varphi}(t)}{\|\widehat{\varphi}(t)\|_{L^2(\omega)}}, \quad a.e.\ t \in (0, T(M)). \tag{2.3}$$

The geometric explanations of Theorem 2.2

Now, we give the geometric explanations of Theorem 2.2. Write $\{S(t,s)\}$ for the evolution system generated by $\Delta - aI$ in $L^2(\Omega)$. Let

$$A_T \triangleq \left\{ \int_0^T S(T,t)\chi_\omega u(t)\,dt \;\middle|\; u \in L^\infty(0,T;L^2(\Omega)) \right\}$$

equipped with the norm

$$\|w\|_{A_T} = \inf\left\{ \|v\|_{L^\infty(0,T;L^2(\Omega))} :\; w = \int_0^T S(T-t)\chi_\omega v(t)\,dt \right\}.$$

Write

$$A_T^M \triangleq \left\{ \int_0^T S(T,t)\chi_\omega u(t) \;\middle|\; u \in \mathcal{U}^M \right\}.$$

It is the ball (in A_T), centered at the origin and of radius M.

The geometric explanations of Theorem 2.2

Theorem 2.3 (G.Wang, Y.Xu, Y. Zhang, 2014)

There is a linear isometry $G : A_T \to X_T^*$ *such that*

$$\langle G(w), z\rangle_{X_T^*, X_T} = \langle w, z\rangle_{L^2(\Omega), L^2(\Omega)}, \quad \forall z \in L^2(\Omega)$$

and such that for each $z \in X_T$,

$$\langle G(w), z\rangle_{X_T^*, X_T} = \int_0^T \langle \hat{v}(t), \chi_\omega \varphi(t; z)\rangle_{L^2(\Omega), L^2(\Omega)}\, dt,$$

where $\hat{v}(\cdot)$ *verifies*

$$w = \int_0^T S(T,t)\chi_\omega \hat{v}(t)\, dt.$$

The geometric explanations of Theorem 2.2

With the aid of Theorem 2.3, we can explain Theorem 2.2 (in particular (2.3)) as follows: $\widehat{\varphi}_{T(M)}$ is a normal vector of the hyperplane which separates the set

$$\{e^{T(M)\Delta}y_0\} + A^M_{T(M)}$$

from the target set $\{0\}$ in X_T^*. (In general, one can only expect that the normal vector belongs to X_T^{**}. But in our case, $\widehat{\varphi}_{T(M)} \in X_T$.) Furthermore, for each $t \in (0, T(M))$, $\widehat{\varphi}(t)$ is a normal vector of the hyperplane separating the set

$$\{e^{t\Delta}y_0\} + A^M_t$$

from the set $\{y(t; u^*)\}$ in $L^2(\Omega)$.

The development of the B-B-P

(i) The B-B-P was first studied by H. Fattorini (1964). He obtained this property for $(TP)^M$ when $\omega = \Omega$. He also realized that

the B-B-P $\Rightarrow$ the uniqueness of time optimal controls.

He proved the B-B-P by contradiction: Suppose that the B-B-P did not hold. Then he would construct a control with an explicit expression such that the solution, associated with this control, reaches the target at time before the minimal time. In his case, the control operator is I. He used the property that $e^{\Delta t} I = I e^{\Delta t}$.
In the case that ω is a proper subset of Ω, it seems that the B-B-P cannot be implied by Fattorini's method. (At least, we do not know how to use his method to prove it.)

The development of the B-B-P

(ii) V. J. Mizel and T. I. Seidman (1997) observed that when $a = a(x)$,

E-observability estimates $\Longrightarrow$ the B-B-P.

E-observability estimates Let $T > 0$ and $E \subset (0, T)$ be of positive measure. Then

$$\Big(\int_\Omega |\varphi(x,0)|^2\,dx\Big)^{\frac{1}{2}} \leq C(\Omega,\omega,T,E)\int_E \Big(\int_\omega |\varphi(x,t)|^2\,dx\Big)^{\frac{1}{2}}\,dt, \tag{3.1}$$

for any φ solves

$$\begin{cases} \varphi_t + \Delta\varphi - a\varphi = 0 & \text{in } \Omega \times (0,T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(\cdot, T) \in L^2(\Omega). \end{cases}$$

The development of the B-B-P

The way to show it is as follows: Assume that an optimal control u^* did not have the B-B-P. Then we would be able to find a $\delta > 0$ and a control $v_\delta \in \mathcal{U}^M$ such that $y(T(M) - \delta; v_\delta) = 0$, which leads to a contradiction to the optimality of $T(M)$. Here, two facts are used: the controlled equation is time-invariant; and the E-controllability holds.

However, it seems for us that this method does not work for the case where the controlled equation is time-varying. For the case where $a = a(x,t)$, we do not know how to get the B-B-P from the E-controllability. If the B-B-P holds is still open, though the E-controllability has been established.

The development of the B-B-P

(iii) When $a = a(x,t)$, we made the following observation: Recall $X_T = \overline{L^2(\Omega)}^{\|\cdot\|_{X_T}}$ with $\|z\|_{X_T} \triangleq \int_0^T \|\chi_\omega \varphi(t;z)\|_{L^2(\Omega)}\, dt, \quad z \in L^2(\Omega)$. Define

$$Y_T = \{z \mid \varphi(\cdot;z) \text{ solves } (2.1), \chi_\omega \varphi(\cdot;z) \in L^1(0,T;L^2(\omega))\}.$$

Theorem 3.1 (G.Wang, Y.Xu, Y. Zhang, 2014)

Suppose that $X_T = Y_T$. Then $(TP)^M$, with $M > \hat{N}$, has the B-B-P.

Recall that when $M \leq \hat{N}$, $(TP)^M$ has no optimal control.

The development of the B-B-P

The idea is to show the above theorem is as follows. Step 1: We show that when $M = N(T)$ for some $T > 0$, where

$$N(T) \triangleq \min\{\|u\|_{L^\infty(0,T;L^2(\Omega))} \ : \ y(T; y_0, u) = 0\},$$

the problem $(TP)^M$ has the B-B-P; Step 2: We show that the function $N(\cdot)$ of T is strictly decreasing, and right continuous and satisfies $\lim_{T\to 0} = \infty$ and $\lim_{T\to\infty} \triangleq \hat{N}$; Step 3: We show that when $X_T = Y_T$, the function $N(\cdot)$ is left continuous.

Here, we used the E-controllability.

When $a = a(x,t)$, we do not know if $X_T = Y_T$. However we can prove it for the case that $a(x,t) = a_1(t) + a_2(x)$ (G. Wang, Y. Xu and Y. Zhang, 2014). Consequently, when $a = a_1(x) + a_2(t)$, (TP^M) has the B-B-P.

Observability from measurable sets in time

(i) When $a \equiv 0$, the E-controllability was obtained by G. Wang (2008). The way is based on the Lebeau-Robbiano spectral inequality (1995)

(ii) When $a = a(x,t)$, the E-observability was built up by K. D. Phung and G. Wang (2013). The way is as follows:
First, based on the frequency function method (L. Escauriaza, F. J. Fernôles and S. Vessella, 2006, and C. C. Poon, 1996), we built up the following unique continuation interpolation inequality:

Observability from measurable sets in time

Given $T > 0$ and an open ball $B_r \subset \Omega$, there are $C_1 = C_1(\Omega, n)$, $C_2 = C_2(T, \|a\|_\infty) > 0$ and $\alpha = \alpha(r, T) \in (0, 1)$ such that

$$\int_\Omega |\varphi(x, L)|^2 \, dx \leq C_1 \Big(\int_{B_r} |\varphi(x, L)|^2 \, dx \Big)^\alpha \times \Big(e^{C_2(1+\frac{1}{T-L})} \int_\Omega |\varphi(x, T)|^2 \, dx \Big)^{1-\alpha}, \tag{4.1}$$

for any $L \in [0, T)$ and any φ solving

$$\begin{cases} \varphi_t + \Delta\varphi - a\varphi = 0 & \text{in } \ \Omega \times (0, T), \\ \varphi = 0 & \text{on } \ \partial\Omega \times (0, T), \\ \varphi(\cdot, T) \in L^2(\Omega). \end{cases}$$

Observability from measurable sets in time

Then, we observed that

the unique continuation estimate (4.1) + the telescoping series method
$\Longrightarrow$ E-observability estimate (3.1).

The above-mentioned telescoping method was partially inspired by the work of L. Miller, 2010.

Observability from measurable sets in time

• It is worth mentioning that in the first work to show the E-observability estimate (3.1), we need the condition that Ω is convex. Then, this result was extended to the case that Ω is of C^2 by K. D. Phung, L. Wang and C. Zhang, 2013.

• When the controlled system is time-varying, with the aid of the E-observability estimate (3.1), we can get the bang-bang property for $(NP)^T$, BUT NOT the B-B-P for $(TP)^M$.

Observability from measurable sets in time

(iii) An abstract framework (time-invariant): Let H be a Hilbert space. Consider the equation

$$\begin{cases} -\varphi_t = A^*\varphi, \quad t > 0, \\ \varphi(T) = \varphi_T. \end{cases} \tag{4.2}$$

We assume $(\mathcal{H}_1)$ $A^* : D(A^*) \subset H \to H$ generates an analytic semigroup in H; $(\mathcal{H}_2)$ there are positive constants C and k such that

$$(O) \qquad \|\varphi(0)\|_H^2 \leq e^{\frac{C}{L^k}} \int_0^L \|B^*\varphi(t)\|_U^2 \, dt, \;\; \forall \varphi_T \in H, \; L \in (0,1]$$

where $B \in \mathcal{L}(U, \mathcal{D}(A^*)')$, with U another Hilbert space.

Observability from measurable sets in time

Theorem 4.1 (G.Wang, C.Zhang, 2014)

Suppose that $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. Then given* $T > 0$ *and* $E \subset (0, T)$ *with* $|E| > 0$*, there exists a constant* $C = C(E, T, k)$ *such that*

$$\|\varphi(0)\|_H \leq C \int_E \|B^*\varphi(t)\|_U \, dt, \ \forall \varphi_T \in H.$$

The way to get it: A combination of the estimate (O), the telescoping series method, and the following propagation of smallness inequality from measurable subsets (for analytic functions) established by S. Vessella, 1999:

Observability from measurable sets in time

Theorem 4.2 (S. Vessella, 1999)

Assume that $f : B_{2R} \subset \mathbb{R}^n \to \mathbb{R}$ *is real analytic in* B_{2R} *verifying*

$$|\partial^\alpha f(x)| \leq \frac{M|\alpha|!}{(\rho R)^{|\alpha|}}, \quad \text{when } x \in B_{2R}, \alpha \in \mathbb{N}^n,$$

for some $M > 0$, $\rho \in (0,1]$. *Let* $E \subset B_R$ *be a measurable set with* $|E| > 0$. *Then there are* $N = N(\rho, |E|/|B_R|)$ *and* $\theta = \theta(\rho, |E|/|B_R|)$ *such that*

$$\|f\|_{L^\infty(B_R)} \leq N\Big(\frac{1}{|E|}\int_E |f|dx\Big)^\theta M^{1-\theta}.$$

Observability from measurable sets in time

Theorem 4.1 can be applied to get the null controllability from measurable sets in time for the following cases:
(i) The time-invariant boundary controlled heat equations. (Here, we used the observability built up by AV. Fursikov and OY Imanuvilov, 1995).
(ii) The stoke equation. (Here we used the observability built up by J.M.Coron and S.Guerrero (2009)).
(iii) The parabolic equation associated with the Grushin operator in rectangles. (Here we used the observability built up by K.Beauchard, P.Cannarsa and R.Guglielmi (2013)).

Observability from measurable sets, time and space

(a) We first introduce the $\mathcal{D}$-observability estimate (i.e., the observability from measurable sets in $\Omega \times (0,T)$). Let $\mathcal{D} \subset \Omega \times (0,T)$ be of positive measure. Then there exists a constant $C = C(\Omega, \mathcal{D}, T)$ such that

$$\left(\int_{\Omega} |\varphi(x,0)|^2 \, dx \right)^{\frac{1}{2}} \leq C \int_{\mathcal{D}} |\varphi(x,t)| \, dxdt, \tag{4.3}$$

for any φ solves

$$\begin{cases} \varphi_t + \Delta\varphi - a\varphi = 0 & \text{in } \Omega \times (0,T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(\cdot, T) \in L^2(\Omega). \end{cases}$$

Observability from measurable sets, time and space

When $a = a(x)$, the $\mathcal{D}$-observability estimate implies the B-B-P for the time optimal control problem:

$$(TOCP)^M \qquad T(M) \triangleq \inf_{u \in \mathcal{V}^M} \{t : \ y(t;u) = 0\},$$

where y is the solution of (1.1) and

$$\mathcal{V}^M = \{u \in L^\infty(\Omega \times \mathbb{R}^+) \mid |u(x,t)| \le M \ \text{ for a.e. } \ (x,t) \in \Omega \times \mathbb{R}^+\}.$$

$(TOCP)^M$ has the B-B-P if any optimal control u^* satisfies that $|u(x,t)| = M$ for a.e. $(x,t) \in \Omega \times (0, T(M))$.

Observability from measurable sets, time and space

When $a \equiv 0$, J. Apraiz, L. Escauriaza, G. Wang and C. Zhang (2013) built up the $\mathcal{D}$-observability. The idea is as follows. Step 1: Using the Vessella inequality, J. Apraiz, L. Escauriaza (2012) built up an observability estimate from measurable set in Ω. Step 2: With the aid of the Lebeau-Robbiano spectral inequality, we built up an interpolation unique continuation inequality. Step 3: Combining these, as well as the telescoping series method, we approach the aim.

Observability from measurable sets, time and space

(b) We next introduce the $\mathcal{J}$-observability estimate: Let $\mathcal{J} \subset \partial\Omega \times (0, T)$ be of positive surface measure. Then there exists a constant $C = C(\Omega, \Gamma, \mathcal{J}, T)$ such that

$$\Big(\int_\Omega |\varphi(x,0)|^2 \, dx \Big)^{\frac{1}{2}} \leq C \int_{\mathcal{J}} \Big| \frac{\partial \varphi}{\partial \nu}(x,t) \Big| \, d\sigma dt,$$

for any φ solves

$$\begin{cases} \varphi_t + \Delta\varphi - a\varphi = 0 & \text{in } \Omega \times (0,T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(\cdot, T) \in L^2(\Omega). \end{cases}$$

Observability from measurable sets, time and space

When Ω is a rectangle and $a = 0$, S. Micu, I. Roventa and M. Tucsnak (2012) built up the $\mathcal{J}$-observability. When Ω is a ball and $a = 0$, S. Micu and L. E. Temeranc$\breve{a}$ (2013) proved this estimate.

For the case where $a = 0$, J. Apraiz, L. Escauriaza, G. Wang and C. Zhang (2013) proved the following:
Write $\Delta_R(q_0) \triangleq B_R(q_0) \cap \partial\Omega$. Suppose that $q_0 \in \partial\Omega$ and $\Delta_{4R}(q_0)$ is real analytic. Then for any $\mathcal{J} \subset \Delta_R(q_0) \times (0, T)$, the $\mathcal{J}$-observability holds.

The method is as follows: we use some Carleman inequality to get an unique continuation interpolation inequality; then apply the Vessella inequality, as well as the telescoping series method to approach the aim.

Recently, L. Escauriaza, Santiago Montaner and C. Zhang (2014) proved both $\mathcal{D}$-observability and $\mathcal{J}$-observability for more general cases. The controlled system reads: $y' + Ay = Bu$. It can be either a parabolic equation associated to the m-th $(m \geq 1)$ power of the Laplace operator, or a parabolic systems associated to the second order elliptic systems. It is assumed that $\partial\Omega$ is analytic, all coefficients in the operator A are real analytic in (x, t). The operator A is not necessary to be self-adjoint.

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Thank you !