Structure of Dynamical Flow and Nonlocal Feedback stabilization for Normal Parabolic equations

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Outline

1) Introduction of Normal Parabolic System (NPS).

2) Basic properties of NPS.

3) Structure of dynamical flow corresponding to NPS.

4) Feedback stabilization of NPS.

5) On possible application of NPS to stabilize equations of the Navier-Stokes type.

Navier-Stokes Equations (NSE)

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \nabla p(t,x) = 0,$$

$$\mathrm{div} v = 0,$$

$$v(t,\dots,x_i,\dots) = v(t,\dots,x_i{+}2\pi,\dots),\ i = 1,2,3,$$

$$v(t,x)|_{t=0} = v_0(x)$$

Here $v(t,x) = (v_1, v_2, v_3)$ is a fluid velocity, $p(t,x)$ is a pressure.

Energy inequality:

$$\int_{\mathbb{T}^3} |v(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{T}^3} |\nabla_x v(\tau,x)|^2 dx d\tau \leq$$

$$\int_{\mathbb{T}^3} |v_0(x)|^2 dx$$

Image of nonlinear operator $(v,\nabla)v$ at each point $v \in \Sigma \equiv \{u \in L_2 : \|u\|_{L_2} = 1\}$ is tangent to the sphere Σ, i.e. $v \perp_{L_2} (v,\nabla)v$

Helmholtz Equations

Curl of velocity

$$\omega(t,x) = \text{curl } v(t,x) =$$

$$= (\partial_{x_2} v_3 - \partial_{x_3} v_2,\ \partial_{x_3} v_1 - \partial_{x_1} v_3,\ \partial_{x_1} v_2 - \partial_{x_2} v_1)$$

Well-known formulas

$$(v,\nabla)v = \omega \times v + \nabla \frac{|v|^2}{2},$$

$$\text{curl } (\omega \times v) = (v,\nabla)\omega - (\omega,\nabla)v, \text{ if } \operatorname{div} v = \operatorname{div} \omega = 0$$

System of equations for curl

$$\partial_t \omega(t,x) - \Delta\omega + (v,\nabla)\omega - (\omega,\nabla)v = 0$$

$$\omega(t,x)|_{t=0} = \omega_0(x)$$

where $\omega_0 = \text{curl} v_0$

System of normal type and its derivation

Function spaces

$$V^m = V^m(\mathbb{T}^3) =$$

$$= \{v(x) \in (H^m(\mathbb{T}^3))^3 : \operatorname{div} v = 0,\ \int_{\mathbb{T}^3} v(x)dx = 0\}$$

where $H^m(\mathbb{T}^3)$ - is the Sobolev space. Using decomposition in Fourier series

$$v(x) = \sum_{k\in\mathbb{Z}^3} \widehat{v}(k)e^{ix\cdot k}, \quad \widehat{v}(k) = \int_{\mathbb{T}^3} \frac{v(x)}{(2\pi)^{-3}} e^{-ix\cdot k}dx,$$

where $x \cdot k = \sum_{j=1}^3 x_j k_j,\ k = (k_1, k_2, k_3)$ and the formula curl curl $v = -\Delta v$, when $\operatorname{div} v = 0$, we get

$$\operatorname{curl}^{-1}\omega(x) = i \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{k \times \widehat{\omega}(k)}{|k|^2} e^{ix\cdot k}$$

Therefore operator

$$\operatorname{curl} \ : \ V^1 \longrightarrow V^0$$

realizes isomorphism of the spaces.

Nonlinear term in Helmholtz equations

$$B(\omega) = (v, \nabla)\omega - (\omega, \nabla)v$$

The following formula holds

$$(B(\omega), \omega)_{V^0} = -\int_{\mathbb{T}^3} \sum_{j,k=1}^{3} \omega_j \partial_j v_k \omega_k dx \neq 0$$

and therefore

$$B(\omega) = B_n(\omega) + B_\tau(\omega),$$

where $B_n(\omega)$ is the component orthogonal to the sphere

$$\Sigma_\omega = \{u \in V^0 : \|u\|_{V^0} = \|\omega\|_{V^0}\}$$

at the point ω, and the vector $B_\tau(\omega)$ is tangent to Σ_ω at ω. It is clear that $B_n(\omega) = \Phi(\omega)\omega$ where Φ is unknown functional, that is determined from equation

$$\int_{\mathbb{T}^3} \Phi(\omega)\omega(x)\cdot\omega(x)dx = \int_{\mathbb{T}^3} (\omega(x), \nabla)v(x)\cdot\omega(x)dx$$

and has the form

$$\Phi(\omega) = \frac{\int_{\mathbb{T}^3}(\omega(x), \nabla)\text{curl }^{-1}\omega(x)\cdot\omega(x)dx}{\int_{\mathbb{T}^3}|\omega(x)|^2dx}, \ \omega \neq 0,$$

$$\Phi(\omega) = 0, \qquad \omega \equiv 0$$

Normal parabolic system (NPS)

$$\partial_t \omega(t,x) - \Delta\omega - \Phi(\omega)\omega = 0, \quad \text{div}\,\omega = 0 \quad (1)$$

$$\omega(t,x)|_{t=0} = \omega_0(x) \qquad (2)$$

Exact formula for NPS solution

Theorem 1. Let $S(t,x,y_0)$ - be solving operator for the Stokes system with periodic boundary conditions:

$$\partial_t y - \Delta y = 0, \ \text{div}\, y = 0, \ y|_{t=0} = y_0, \qquad (3)$$

i.e. $S(t,x,y_0) = y(t,x)$. (We assume that $\text{div}\, y_0 = 0$). Then solution of the problem (1),(2) has the form

$$\omega(t,x;\omega_0) = \frac{S(t,x;\omega_0)}{1 - \int_0^t \Phi(S(\tau,x;\omega_0))d\tau} \qquad (4)$$

Unique solvability of NPS and continuity of solutions on initial conditions

Lemma 1. $\exists c > 0,\ \forall u \in V^{3/2} \quad \Phi(u) \le c\|u\|_{3/2}$
Lemma 2. $\forall \beta < 1/2 \quad \exists c_1 > 0 \quad \forall y_0 \in V^{-\beta}(\mathbb{T}^3)$,

$$\int_0^t \Phi(S(t,\cdot,y_0))dt \le c_1\|y_0\|_{-\beta}$$

Let $Q_T = (0,T)\times\mathbb{T}^3$, $T > 0$ or $T = \infty$. The space of solutions for NPS:

$$V^{1,2(-1)}(Q_T) = L_2(0,T;V^1)\cap H^1(0,T;V^{-1})$$

Moreover, we look for solutions $\omega(t,x;\omega_0)$ satisfying

Condition 1. If initial condition $\omega_0 \in V^0\setminus\{0\}$ and solution $\omega(t,x;\omega_0) \in V^{1,2(-1)}(Q_T)$ then $\omega(t,\cdot,\omega_0) \ne 0\ \forall t \in [0,T]$

Theorem 2. For each $\omega_0 \in V^0$ there exists $T > 0$ such that there exists unique solution $\omega(t,x;\omega_0) \in V^{1,2(-1)}(Q_T)$ of the problem (1),(2) satisfying Condition 1.

Theorem 3. The solution $\omega(t,x;\omega_0) \in V^{1,2(-1)}(Q_T)$ of the problem (1),(2) depends continuously on initial condition $\omega_0 \in V^0$.

On kernel of the functional $\Phi(S(t;u))$

Define the cone

$$K\Phi = \{u \in V^0 : \ \Phi(S(t;u)) \equiv 0 \quad \forall t \in \mathbb{R}_+\}$$

If $u \in K\Phi$ then $\lambda u \in K\Phi \ \ \forall \lambda \in \mathbb{R}$

Let

$$L = \{z \in V^0 : \ z(x) = \sum_{k \in \mathcal{U}} \hat{z}(k) e^{ik\cdot x}, \ \hat{z}(-k) = \overline{\hat{z}(k)}\},$$

where

$$\mathcal{U} = \{k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \backslash \{0\} : \ \sum_{j=1}^{3} k_j \text{ is odd}\}$$

Lemma 3. $L \subset K\Phi, \quad K\Phi \setminus L \neq \emptyset$

Structure of dynamical flow for NPS

$V^0(\mathbb{T}^3) \equiv V^0$ is phase space for problem (1),(2).

Definition 1. The set $M_- \subset V^0$ of ω_0, such that for solution $\omega(t,x;\omega_0)$ of problem (1),(2) satisfies inequality

$$\|\omega(t,\cdot;\omega_0)\|_0 \le \alpha \|\omega_0\|_0 e^{-t/2} \qquad \forall t > 0 \quad (*)$$

is called the set of stability. Here $\alpha > 1$ is a fixed number depending on $\|\omega_0\|_0$.

$$M_-(\alpha) = \{\omega_0 \in M_-;\ \omega(t,\cdot;\omega_0) \text{ satisfies } (*)\}$$

where $\alpha \ge 1$ is fixed. Then $M_- = \cup_{\alpha \ge 1} M_-(\alpha)$

If for $\omega_0 \in V^0$ the bound

$$\sup_{t \in \mathbb{R}_+} \int_0^t \Phi(S(\tau,\cdot;\omega_0))d\tau \le \frac{\alpha - 1}{\alpha}$$

holds then $\omega_0 \in M_-(\alpha)$.

Definition 2. The set $M_+ \subset V^0$ of ω_0, such that the corresponding solution $\omega(t, x; \omega_0)$ exists only on a finite time interval $t \in (0, t_0)$, and blows up at $t = t_0$ is called the set of explosions.

The formula holds:

$$M_+ = \{\omega_0 \in V^0 : \ \exists t_0 > 0 \ \int_0^{t_0} \Phi(S(\tau, \cdot; \omega_0))d\tau = 1\}$$

Definition 3. The set $M_g \subset V^0$ of ω_0, such that the corresponding solution $\omega(t, x; \omega_0)$ exists for time $t \in \mathbb{R}_+$, and $\|\omega(t, x; \omega_0)\|_0 \to \infty$ as $t \to \infty$ is called the set of growing.

Lemma 4. Sets M_-, M_+, M_g are not empty, and $M_- \cup M_+ \cup M_g = V^0$

Some subsets of unit sphere from V^0

Unit sphere: $\Sigma = \{v \in V^0 : \|v\|_0 = 1\}$. Subsets

$$A_-(t) = \{v \in \Sigma : \int_0^t \Phi(S(\tau, v))d\tau \leq 0\},$$

$$A_0(t) = \{v \in \Sigma : \int_0^t \Phi(S(\tau, v))d\tau = 0\}$$

$$A_- = \cap_{t\geq 0} A_-(t),\ A_0 = \cap_{t\geq 0} A_0(t)$$

$$B_+ = \Sigma \setminus A_- \equiv$$

$$\equiv \{v \in \Sigma :\ \exists t_0 > 0\ \int_0^{t_0} \Phi(S(\tau, v))d\tau > 0\},$$

$$\partial B_+ = \{v \in \Sigma :\ \forall t > 0 \int_0^t \Phi(S(\tau, v))d\tau \leq 0$$

$$\text{и } \exists t_0 > 0 :\ \int_0^{t_0} \Phi(S(\tau, v))d\tau = 0\}$$

On a structure of phase space

Important function on sphere Σ:

$$B_+ \ni v \to b(v) = \max_{t \geq 0} \int_0^t \Phi(S(\tau, v)) d\tau \quad (5)$$

Evidently, $b(v) > 0$ и $b(v) \to 0$ as $v \to \partial B_+$.
Let define the map $\Gamma(v)$:

$$B_+ \ni v \to \Gamma(v) = \frac{1}{b(v)} v \in V^0 \quad (6)$$

It is clear that $\|\Gamma(v)\|_0 \to \infty$ as $v \to \partial B_+$.
The set $\Gamma(B_+)$ divides V^0 on two parts:

$$V_-^0 = \{v \in V^0 : \ [0, v] \cap \Gamma(B_+) = \emptyset\},$$

$$V_+^0 = \{v \in V^0 : \ [0, v) \cap \Gamma(B_+) \neq \emptyset\}$$

Let $B_+ = B_{+,f} \cup B_{+,\infty}$ where

$$B_{+,f} = \{v \in B_+ : \text{ max in (5) achives at } t < \infty\}$$

$$B_{+,\infty} = \{v \in B_+ : \text{ max in (5) does not achive at } t < \infty\}$$

Theorem 4. $M_- = V_-^0$, $M_+ = V_+^0 \cup B_{+,f}$, $M_g = B_{+,\infty}$

Burgers equation

$$\partial_t y(t,x) - \partial_{xx} y - \partial_x y^2 = 0,\ x \in (-\pi,\pi), \quad (7)$$

$$y(t,x+2\pi) = y(t,x),\ y|_{t=0} = y_0(x), \quad (8)$$

considered in phase space

$$Y^1 = \{y_0 \in H^1(-\pi,\pi) : \int_{-\pi}^{\pi} y_0(x)dx = 0\},$$

where $\|y\|_{Y^1} = \|y_x\|_{L_2}$.

Nonlinearity of normal type

Differentiation (7) on x yields

$$\partial_t v - \partial_{xx} v - B(y) = 0, \quad B(y) = 2v^2 + 2yv_x$$

where $v = \partial_x y$. Let us decompose

$$B(y) = B_n(y) + B_\tau(y),$$

where $B_n(y) \perp \Sigma(Y^1), B_\tau(y)$ touches $\Sigma(Y^1)$ and $\Sigma(Y^1) = \{y \in Y^1 : \|y\|_{Y^1} = 1\}$ Then

$$B_n(y) = \Phi(y_x)y_x, \quad \Phi(v) = \frac{\int_{-\pi}^{\pi} v^3 dx}{\int_{-\pi}^{\pi} v^2 dx}$$

Equation with normal nonlinearity

$$\partial_t v - \partial_{xx} v - \Phi(v)v = 0, \qquad (9)$$

$$v(t, x + 2\pi) = v(t, x), \quad v|_{t=0} = v_0(x) \quad (10)$$

Phase space:

$$L_2^0 = \{v \in L_2(-\pi, \pi) : \int_{-\pi}^{\pi} v(x)dx = 0\}$$

Definition 1. *The set $M_- \subset L_2^0$, of all initial conditions v_0 for problem (9),(10) whose solutions satisfy*

$$\|v(v, \cdot)\|_{L_2}^2 \leq \alpha e^{-t}$$

with a certain $\alpha = \alpha(v_0) > 0$ is called set of stability.

Definition 2. *The set $M_+ \subset L_2^0$ of all initial conditions v_0 for problem (9),(10) whose solutions blow up during finite time is called the set of explosions.*

Definition 3. *The set $M_g = L_2^0 \setminus (M_- \cup M_+)$ is called the set of grouth.*

Denote $S(t, x, v_0) = w(t, x)$ where w is the solution of the problem

$$\partial_t w - \partial_{xx} w = 0,$$

$$w(t, x + 2\pi) = w(t, x), \quad w|_{t=0} = v_0(x)$$

Formula for solution of (9),(10):

$$v(t, x, v_0) = \frac{S(t, x, v_0)}{1 - \int_0^t \Phi(S(\tau, \cdot, v_0))d\tau} \qquad (11)$$

Lemma 1. $M_- \neq \emptyset,\ M_+ \neq \emptyset,\ M_g \neq \emptyset.$

Lemma 2. For initial conditions $v_0 \in M_g$ the solution $v(t, x, v_0)$ of problem (9),(10) with normal nonlinearity satisfies

$$\|v(t, \cdot, v_0)\|_{L_2} \to \infty \quad \text{as} \quad t \to \infty$$

Feedback stabilization of equation with normal nonlinearity.

We consider stabilization problem

$$\partial_t v - \partial_{xx} v - \Phi(v)v = 0, \quad v|_{t=0} = v_0(x) + u_0(x)$$

on circumference, where $v_0(x)$ is a given function and $u_0(x)$ is a starting control supported on a segment $[-\rho, \rho] \subset [-\pi, \pi]$ with arbitrary prescribed $\rho > 0$.

We look for universal stabilizing control

$$u_0(x) = \lambda u(x), \quad \lambda \in \mathbf{R} \qquad (12)$$

with

$$u(x) = \xi_p(x)(\cos 2px + \cos 4px) \qquad (13)$$

where p is a natural number satisfying $\pi/(2p) \leq \rho$, and $\xi_p(x)$ is characteristic function of segment $[-\pi/(2p), \pi/(2p)]$.

Theorem 5. Given $v_0 \in M_+ \cup M_g$, $\rho > 0$ is small and fixed. There exists $u_0 \in L_2^0$ of the form (12), (13) such that $v_0 + u_0 \in M_-$.

The main step of proof consists of establishing inequality

$$\int_{-\pi}^{\pi} S^3(t,x,u)dx \geq \beta e^{-6t} \qquad (14)$$

with a positive β where $S(t,x,u)$ is the solution of heat equation with periodic boundary condition and initial condition $u(x)$ defined in (13).

Using (14) it is possible to prove that

$$\forall v_0 \in M_+ \cup M_g \quad \exists \alpha > 1,\ \hat{\lambda} \gg 1 \quad \forall |\lambda| \geq \lambda_0$$

$$1 - \int_0^t \Phi(S(t,x,v_0+\lambda u)dx \geq 1/\alpha \qquad (15)$$

In virtue of explicit formula (11) for solution of NPE (15) implies that

$$\|v(t,\cdot;v_0+\lambda u)\|_{L_2}^2 \leq \alpha e^{-t}$$

This proves Theorem 5.

Remark 1 Using result obtained in the Theorem 5 one can prove non local stabilization of differentiated Burgers equation:

$$\partial_t v - \partial_{xx} v - \Phi(v)v = B_\tau(y) \equiv 2v^2 + 2yv_x - \Phi(v)v$$

We apply starting control as for $B_\tau(v) = 0$:

$$v|_{t=0} = v_0(x) + \lambda_0 u(x)$$

with $\lambda_0 = \lambda$. If $\exists t_1 > 0$ such that $v(t_1, \cdot) \in M_g \cup M_+$ we apply starting control again:

$$v|_{t=t_1} = v(t_1, x) + \lambda_1 u(x)$$

with a proper λ_1 ($|\lambda_1| \leq |\lambda|$). In fact we get stabilization of differentiated Burgers equations by feedback impulse control:

$$\partial_t v - \partial_{xx} v - \Phi(v)v = B_\tau(v) + \sum_{j=0}^{N} \lambda_j u(x)\delta(t - t_j),$$

$$v|_{t=0} = v_0(x)$$

where $t_0 = 0$, λ_j are chosen satisfying $|\lambda_j| \leq |\lambda_0|$ and t_j are defined by the rule indicated above.

Remark 2 In the case of local feedback stabilization problem it is well-known connection between starting, impulse, an distributed control that are concentrated in a space sub domain, i.e. it is known how to construct impulse control with help of starting one, and how to transform impulse control to the distributed one. (see, for instance,A.V.Fursikov, A.V.Gorshkov. Certain questions of feedback stabilization for Navier-Stokes equations.- Evolution Equations and Control Theory, v.1(1), 2012, p.109-140) Similar results can be obtained in the case of nonlocal stabilization for NPS. Hence Theorem 5 implies possibility of nonlocal stabilization for NPS by impulse and by distributed feedback controls.

Taking into account Remark 1 this opens opportunity to construct nonlocal stabilization for equations of Navier-Stokes type by impulse and by distributed feedback controls.

This talk is based on the following papers:

1)A.V.Fursikov The simplest semilinear paraboloic equation of normal type.-Mathematical Control and Related Fields, v.2(2), 2012

2)A.V.Fursikov On the parabolic system of normal type corresponding to 3D Helmholds sytem (Accepted for publication)

3)A.V.Fursikov Stabilization of the simplest normal parabolic equation by the starting control (Accepted for publication)

Thank you

for attention