On the controllability of viscous fluid flows with non-constant densities

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Outline

Outline

The setting

Consider the 1-d compressible Navier Stokes equation:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{in } (0,T)\times(0,L), \\ \rho\,(\partial_t u + u\partial_x u) - \nu\partial_{xx} u + \partial_x p(\rho) = 0, & \text{in } (0,T)\times(0,L). \end{cases}$$

- ρ is the density;
- u is the velocity;
- $p(\rho)$ is the pressure, assumed to satisfy $p'(\rho) > 0$, typically $p(\rho) = \alpha\rho^\gamma$, $\alpha > 0$, $\gamma \geq 1$;
- $\nu > 0$ is the viscosity (constant).

The controls

The boundary conditions are not prescribed. They will be the controls, e.g.

$$u(t,0) = v_0(t), \quad u(t,L) = v_L(t),$$

and

$$\rho(t,0) = w_0(t) \text{ if } u(t,0) > 0, \quad \rho(t,L) = w_L(t) \text{ if } u(t,L) < 0,$$

These functions (v_0, v_L, w_0, w_L) can be chosen freely.

The control problem

The goal is to understand how one can act on the system to steer it to “a suitable state” at time T.

We focus on the local exact controllability around constant states

$$\bar{\rho} = \text{constant}(> 0), \quad \bar{u} = \text{constant}.$$

The problem is to find $\varepsilon > 0$ such that if

$$\|(\rho_0, u_0) - (\bar{\rho}, \bar{u})\|_* \leq \varepsilon$$

there exist control functions such that the solution (ρ, u) satisfies

$$\rho(T) = \bar{\rho}, \quad u(T) = \bar{u}.$$

Our study relies on the control properties of the linearized system:

$$\begin{cases} \partial_t \rho + \bar{u}\partial_x \rho + \bar{\rho}\partial_x u = 0, & \text{in } (0,T)\times(0,L), \\ \bar{\rho}\left(\partial_t u + \bar{u}\partial_x u\right) - \nu\partial_{xx} u + p'(\bar{\rho})\partial_x \rho = 0, & \text{in } (0,T)\times(0,L). \end{cases}$$

- Transport phenomenon: $L/\bar{u}$ is a critical time.
- Local exact controllability around $(\bar{\rho},\bar{u}) = (1,0)$ fails, see Rosier-Rouchon '07.
- one can construct Gaussian beams type solutions of the adjoint equations traveling at velocity $\bar{u}$ and arbitrarily small on the boundaries, hence contradicting observability in time $T < L/\bar{u}$, see Debayan '14.
- There are strong coupling terms (in blue).

Theorem (S.E., O. Glass, S. Guerrero, J.-P. Puel 2012)

Let $(\bar{\rho}, \bar{u}) \in \mathbb{R}_+^* \times \mathbb{R}^*$ and $T > L/|\bar{u}|$. Then there exists $\varepsilon > 0$ such that for all $(\rho_0, u_0) \in H^3(0, L)^2$ satisfying

$$\|(\rho_0, u_0) - (\bar{\rho}, \bar{u})\|_{(H^3)^2} \leq \varepsilon,$$

there exists a controlled trajectory (ρ, u)

$$\rho \in H^1((0, T) \times (0, L))$$
$$u \in H^1((0, T); L^2(0, L)) \cap L^2((0, T); H^2(0, L))$$

satisfying

$$(\rho(T), u(T)) = (\bar{\rho}, \bar{u}).$$

- As expected, $L/|\bar{u}|$ is the critical time.

Related references

On the Cauchy problem: Many results, cf. P.L. Lions' book.
We used a regularity result of Matsumura-Nishida 1980.
$\rightsquigarrow$ Claim: We may shortcut the proof and get the result for $\|(\rho_0, u_0) - (\bar{\rho}, \bar{u})\|_{(H^1)^2} \leq \varepsilon$.

Controllability results for compressible Navier Stokes equations

- Negative results for linearized 1-d compressible NS equations $\bar{u} = 0$: Rosier-Rouchon (2007), Chowdhury-Ramaswamy-Raymond (2012)
- Local exact controllability to trajectories for 1-d compressible NS equations in Lagrangian coordinates when the initial density coincides with the one of the reference trajectory, Amosova (2011).

Controllability results for Euler equations:

- Exact controllability for incompressible Euler equations: Coron (1996), Glass (2000).
- Exact controllability for 1-d compressible Euler equations: Li-Rao (2003) for classical solutions, Glass (2007) for entropy solutions (exact controllability between neighborhoods of constant states + Oleinik type conditions) .
- Exact controllability for 1-d non-isentropic Euler equations, Glass (2013).

Outline

1. 1-d compressible Navier-Stokes equation
2. General strategy
3. 2d incompressible Navier-Stokes equations

General strategy

We use a fixed point argument:

Given $(\hat\rho, \hat u)$, find a solution (ρ, u) of

$$\begin{cases} \partial_t \rho + \bar u \partial_x \rho + \bar\rho \partial_x u = f(\hat\rho, \hat u), & \text{in } (0,T)\times(0,L), \\ \bar\rho\,(\partial_t u + \bar u \partial_x u) - \nu \partial_{xx} u + p'(\bar\rho)\partial_x \hat\rho = g(\hat\rho, \hat u), & \text{in } (0,T)\times(0,L). \end{cases}$$

satisfying

$$(\rho, u)(t=0) = (\rho_0, u_0), \quad (\rho, u)(t=T) = (0,0).$$

- The map is defined as a solution of a control problem.
- One has to show that the map $(\hat\rho, \hat u) \mapsto (\rho, u)$ maps a ball into itself.

The control strategy for the density

Let us recall the control problem for ρ:

$$\begin{cases} \partial_t \rho + \bar{u}\partial_x \rho + \bar{\rho}\partial_x u = f(\hat{\rho}, \hat{u}), \\ \rho(0,x) = \rho_0(x), \quad \rho(T,x) = 0 \end{cases} \quad \text{on } (0, L).$$

$\rightsquigarrow$ Transport equation at a velocity $\bar{u}$ (>0).

Given u, this is a simple control problem !

The control strategy for the density (2)

With $\tilde{f}(\hat{\rho}, \hat{u}, u) = f(\hat{\rho}, \hat{u}) - \bar{\rho}\partial_x u$, introduce ρ_f, ρ_b the solutions of

$$\begin{cases} \partial_t \rho_f + \bar{u}\partial_x \rho_f = \tilde{f}(\hat{\rho}, \hat{u}, u), & \text{in } (0,T) \times (-\infty, \infty), \\ \rho_f(0,x) = \rho_0(x), \end{cases}$$

and

$$\begin{cases} \partial_t \rho_b + \bar{u}\partial_x \rho_b = \tilde{f}(\hat{\rho}, \hat{u}, u), & \text{in } (0,T) \times (-\infty, \infty), \\ \rho_b(T,x) = 0. \end{cases}$$

- if the time T is large enough, one can glue ρ_f and ρ_b to obtain a solution of the control problem.

The control strategy for the density (3)

Let $\eta = 1$ for $x > b$ ($b < 0$) and $\eta = 0$ for $x < a$ ($a < b$),

$$\rho(t,x) = \rho_f(t,x)\eta(x - \bar{u}t) + \rho_b(t,x)(1 - \eta(x - \bar{u}t))$$

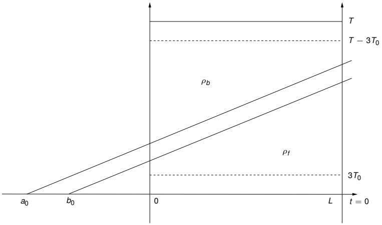

Figure : The straight lines are $t \mapsto (t, a + \overline{u}t)$ and $t \mapsto (t, b + \overline{u}t)$.

Stability with respect to velocity

If one wants to control

$$\begin{cases} \partial_t \rho + (\bar{u} + \tilde{u})\partial_x \rho = \tilde{f}(\hat{\rho}, \hat{u}, u), \\ \rho(0,x) = \rho_0(x), \quad \rho(T,x) = 0 \end{cases} \quad \text{on } (0,L).$$

with $\|\tilde{u}\| \ll |\overline{u}|$, the same strategy applies:
Let X be the flow associated to $u = \bar{u} + \tilde{u}$, i.e.

$$\frac{dX}{dt}(t,s,x) = (\bar{u} + \tilde{u})\,(t, X(t,s,x)), \quad X(s,s,x) = x.$$

Let $\eta = 1$ for $x > b$ ($b < 0$) and $\eta = 0$ for $x < a$ ($a < b$),

$$\rho(t,x) = \rho_f(t,x)\eta(X(0,t,x)) + \rho_b(t,x)(1 - \eta(X(0,t,x))$$

Controlling the velocity

- Heat-type control problem for the velocity.
  $\rightsquigarrow$ We follow the classical strategy of Fursikov Imanuvilov '96 for the control of heat-type equations.

BUT we need estimates:

- Carleman estimates, i.e. observability estimates in suitably weighted spaces.
  AND we should be able to get estimates on the density in these weighted norms.

$\rightsquigarrow$ Weight functions that follow the characteristics $t \mapsto x_0 + \bar{u}t$.

Related results: Albano Tataru '00, Chaves Rosier Zuazua '13

The classical approach for the heat equation

The heat equation controlled through $\omega \subset (0, L)$:

$$\begin{cases} \partial_t u - \partial_{xx} u = g + v(t, x)\chi_\omega, & (t, x) \in (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = 0 & t \in (0, T), \\ u(0, x) = u_0(x). & \end{cases}$$

Control requirement

Find v such that $u(T, x) = 0$

- g is a source term, usually comes from the non-linearity;
- u_0 is the initial data to be controlled;
- v is the control function.

The classical approach for the heat equation

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- $u_0 = 0$;
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Duality and observability

Weak Formulation

For all smooth function z satisfying $z(t,0) = z(t,L) = 0$,

$$\int_0^T \int_0^L u\,(-\partial_t z - \partial_{xx} z) - \int_0^T \int_0^L gz - \int_0^T \int_\omega vz = 0.$$

In particular, if one gets a minimizer Z of

$$J(z) = \frac{1}{2}\int_0^T \int_0^L |-\partial_t z - \partial_{xx} z|^2 + \frac{1}{2}\int_0^T \int_\omega |z|^2 - \int_0^T \int_0^L gz,$$

then $U = (-\partial_t - \Delta)Z$, $V = -Z\chi_\omega$ solves the control problem.
Requires the following observability inequality

$$\int_0^T \int_0^L |z|^2 \lesssim \int_0^T \int_0^L |-\partial_t z - \partial_{xx} z|^2 + \int_0^T \int_\omega |z|^2.$$

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Carleman estimates (Fursikov-Imanuvilov 1996)

There exist $s_0, \lambda_0 > 1$ such that for all $s \geq s_0$ $\lambda \geq \lambda_0$, any smooth z such that $z(t,0) = z(t,L) = 0$ satisfies

$$s^3\lambda^4 \int_0^T \int_0^L \xi^3 e^{-2s\varphi}|z|^2 + s\lambda^2 \int_0^T \int_0^L \xi e^{-2s\varphi}|\partial_x z|^2 + \frac{1}{s}\int_0^T \int_0^L \frac{1}{\xi} e^{-2s\varphi}\left(|\partial_{xx} z|^2 + |\partial_t z|^2\right) \leq C \int_0^T \int_0^L e^{-2s\varphi}\left|-\partial_t z + \partial_{xx} z\right|^2 + Cs^3\lambda^4 \int_0^T \int_\omega \xi^3 e^{-2s\varphi}|z|^2$$

- Weighted observability estimate with weights φ, ξ.
- the powers of s and λ allow to absorb lower order terms $\rightsquigarrow$ quantification of the compactness.

Classical Weight functions

Let $\psi = \psi(x)$, $\psi : [0, L] \mapsto [3, 4]$, with

$$\psi'(x) < 0 \text{ near } x = L \quad \text{and } \psi'(x) > 0 \text{ near } \ x = 0$$

$$\inf_{(0,L)\setminus\omega} |\psi'| > 0.$$

For $s, \lambda \geq 1$, define

$$\varphi(t,x) = \frac{1}{t(T-t)}\left(e^{5\lambda} - e^{\lambda\psi(x)}\right), \quad \xi(t,x) = \frac{e^{\lambda\psi(x)}}{t(T-t)}$$

A modification

Our weight functions defined for $x \in (-3\bar{u}T, L)$

Let $\psi = \psi(x)$, $\psi : [-5\bar{u}T, L] \mapsto [3, 4]$, with

$$\psi'(x) < 0 \text{ near } x = L \quad \text{and } \psi'(x) > 0 \text{ near } x = -3\bar{u}T$$

$$\inf_{(-5\bar{u}T, L)\setminus(-3\bar{u}T, -2\bar{u}T)} |\psi'| > 0.$$

For $s, \lambda \geq 1$, define

$$\varphi(t, x) = \frac{1}{\theta(t)} \left(e^{5\lambda} - e^{\lambda\psi(x-\bar{u}t)} \right), \quad \xi(t, x) = \frac{e^{\lambda\psi(x-\bar{u}t)}}{\theta(t)}$$

with θ such that, for T_0 small,

$$\theta(t) = \begin{cases} t \text{ in } (0, T_0), \\ 1 \text{ in } (2T_0, T - 2T_0) \\ T - t \text{ in } (T - T_0, T) \end{cases} \qquad \begin{cases} \theta' > 0 \text{ in } (T_0, 2T_0), \\ \theta' < 0 \text{ in } (T - 2T_0, T - T_0). \end{cases}$$

Setting $\tilde{L} = -3\bar{u}T$, as before we get

Carleman estimates

There exist $s_0, \lambda_0 > 1$ such that for all $s \geq s_0$ $\lambda \geq \lambda_0$, any smooth z such that $z(t, \tilde{L}) = z(t, L) = 0$ satisfies

$$
\begin{aligned}
s^3\lambda^4 \int_0^T \int_{\tilde{L}}^L \xi^3 e^{-2s\varphi} |z|^2 + s\lambda^2 \int_0^T \int_{\tilde{L}}^L \xi e^{-2s\varphi} |\partial_x z|^2 \\
+ \frac{1}{s} \int_0^T \int_{\tilde{L}}^L \frac{1}{\xi} e^{-2s\varphi} \left(|\partial_{xx} z|^2 + |\partial_t z|^2 \right) \\
\leq C \int_0^T \int_{\tilde{L}}^L e^{-2s\varphi} \left| -\partial_t z + \partial_{xx} z \right|^2 \\
+ C s^3 \lambda^4 \int_0^T \int_{(-3\bar{u}T, -\bar{u}T)} \xi^3 e^{-2s\varphi} |z|^2.
\end{aligned}
$$

Using duality, we find a controlled trajectory u such that $u(t, L) = 0$ and

$$s^3\lambda^4 \int_0^T \int_0^L |u|^2 e^{2s\varphi} + s\lambda^2 \int_0^T \int_0^L \frac{e^{2s\varphi}}{\xi^2} |\partial_x u|^2 + \frac{1}{s} \int_0^T \int_0^L \frac{e^{2s\varphi}}{\xi^4} \left(|\partial_t u|^2 + |\partial_{xx} u|^2 \right) \leq C \int_0^T \int_0^L \frac{e^{2s\varphi}}{\xi^3} |g|^2.$$

The difficult part now is to get estimates of the source term, in particular of $\partial_x \hat{\rho}$

$$\int_0^T \int_0^L \frac{e^{2s\varphi}}{\xi^3} |\partial_x \rho|^2.$$

Strong coupling terms ?

The linearized equations

$$\begin{cases} \partial_t \rho + \bar{u}\partial_x \rho + \bar{\rho}\partial_x u = 0, & \text{in } (0,T)\times(0,L), \\ \bar{\rho}\left(\partial_t u + \bar{u}\partial_x u\right) - \nu \partial_{xx} u + p'(\bar{\rho})\partial_x \rho = 0, & \text{in } (0,T)\times(0,L). \end{cases}$$

should rather be considered in terms of $(\partial_x \rho, u)$:

$$\begin{cases} \partial_t \partial_x \rho + \bar{u}\partial_x(\partial_x \rho) + \bar{\rho}\partial_{xx} u = 0, & \text{in } (0,T)\times(0,L), \\ \bar{\rho}\left(\partial_t u + \bar{u}\partial_x u\right) - \nu \partial_{xx} u + p'(\bar{\rho})\partial_x \rho = 0, & \text{in } (0,T)\times(0,L). \end{cases}$$

Hence

$$\partial_x \rho \sim \partial_{xx} u$$

$\rightsquigarrow$ direct estimates will not allow to conclude...

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Hence

$$\partial_x \rho \sim \partial_{xx} u$$

$\rightsquigarrow$ direct estimates will not allow to conclude...

Strong coupling terms ?

Indeed, if we introduce

$$\mu = u + \frac{\nu}{\bar{\rho}^2} \partial_x \rho,$$

the equation in (μ, u) are:

$$\begin{cases} \bar{\rho}(\partial_t \mu + \bar{u} \partial_x \mu) + \frac{p'(\bar{\rho})\bar{\rho}^2}{\nu} \mu - u = 0, & \text{in } (0, T) \times (0, L), \\ \bar{\rho}\left(\partial_t u + \bar{u} \partial_x u\right) - \nu \partial_{xx} u + \frac{p'(\bar{\rho})\bar{\rho}^2}{\nu} \mu - u = 0, & \text{in } (0, T) \times (0, L). \end{cases}$$

So that

$$\mu \sim u$$

and the coupling appears strictly weaker.

- This allows to derive estimates on $\partial_x \rho$ in the weighted L^2 space.
- Fixed point spaces are given in weighted Carleman norms $(\ldots)$

Remark

The quantity

$$\mu = u + \frac{\nu}{\bar{\rho}^2} \partial_x \rho$$

is the so-called effective velocity, cf Bresch Desjardins 2003.

Further comments

Ongoing works:

- Local exact controllability around smooth trajectories under suitable geometric conditions on the flow of the target trajectory.
- Local exact controllability for compressible Navier-Stokes equations in higher dimension.

Open problems:

- Local exact controllability around $(\bar{\rho}, \bar{u}) = (1, 0)$?
  $\rightsquigarrow$ Coron's return method ?
- Global controllability to trajectories ? ...

Outline

The setting

Let $\Omega \subset \mathbb{R}^2$, $\Omega_T = (0, T) \times \Omega$.
Consider the 2-d non-homogeneous incompressible Navier Stokes equation and a particular trajectory $(\overline{\sigma}, \overline{\mathbf{y}})$:

$$\begin{cases} \partial_t \overline{\sigma} + \operatorname{div}(\overline{\sigma}\, \overline{\mathbf{y}}) = \overline{f_\sigma} & \text{in } \Omega_T, \\ \overline{\sigma}\, (\partial_t \overline{\mathbf{y}} + (\overline{\mathbf{y}} \cdot \nabla)\overline{\mathbf{y}}) - \nu \Delta \overline{\mathbf{y}} + \nabla \overline{q} = \overline{\mathbf{f}_\mathbf{y}} & \text{in } \Omega_T, \\ \operatorname{div} \overline{\mathbf{y}} = 0 & \text{in } \Omega_T, \\ (\overline{\sigma}(0), \overline{\mathbf{y}}(0)) = (\overline{\sigma_0}, \overline{\mathbf{y}_0}) & \text{in } \Omega, \end{cases}$$

- $\overline{\sigma}$ is the density;
- $\overline{\mathbf{y}}$ is the velocity;
- $\overline{q}$ is the pressure;
- $\nu > 0$ is the viscosity (constant).

The problem

We have an error on the initial data:

$$\begin{cases} \partial_t \sigma + \operatorname{div}(\sigma \mathbf{y}) = \overline{f_\sigma} & \text{in } \Omega_T, \\ \sigma\left(\partial_t \mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y}\right) - \nu \Delta \mathbf{y} + \nabla q = \overline{\mathbf{f_y}} & \text{in } \Omega_T, \\ \operatorname{div} \mathbf{y} = 0 & \text{in } \Omega_T, \\ (\sigma(0), \mathbf{y}(0)) = (\overline{\sigma} + \rho_0, \overline{\mathbf{y}} + \mathbf{u}_0), & \text{in } \Omega, \end{cases}$$

Goal

Control the trajectory $(\sigma, \mathbf{y})$ such that

$$(\sigma(T), \mathbf{y}(T)) = (\overline{\sigma}(T), \overline{\mathbf{y}}(T)) \quad \text{in } \Omega.$$

$\rightsquigarrow$ Where are the controls ?

The controls

The controls act on the boundary:

$$\sigma = \overline{\sigma} + h_\sigma \text{ for } (t,x) \in (0,T) \times \partial\Omega, \text{ with } \mathbf{y}(t,x)\cdot\mathbf{n}(x) < 0,$$
$$\mathbf{y} = \overline{\mathbf{y}} + \mathbf{h_y} \text{ on } (0,T) \times \partial\Omega.$$

A more restrictive condition: Supp $\mathbf{h_y} \subset (0,T) \times \Gamma_c$, $\Gamma_c \subset \partial\Omega$,

$\Gamma_0 = \partial\Omega \setminus \Gamma_c$ satisfies

$\exists \gamma > 0$, s.t. $\forall t \in (0,T)$, $\forall x \in \Gamma_0$, $\overline{\mathbf{y}}(t,x)\cdot\mathbf{n}(x) \geq \gamma$

and Γ_c has a finite number of connected components.

Main assumptions

- Regularity of the target trajectory

$$(\overline{\sigma}, \overline{\mathbf{y}}) \in C^2([0,T] \times \overline{\Omega}) \times \mathbf{C}^2([0,T] \times \overline{\Omega}),$$
$$\inf_{[0,T]\times\overline{\Omega}} \overline{\sigma} > 0.$$

- a Geometric Condition: Introducing the flow $\overline{X}$

$$\forall (t,\tau,x) \in [0,T]^2 \times \mathbb{R}^2, \quad \begin{cases} \partial_t \overline{X}(t,\tau,x) = \overline{\mathbf{y}}(t, \overline{X}(t,\tau,x)), \\ \overline{X}(\tau,\tau,x) = x, \end{cases}$$

we assume that

$$\left\{ x \in \overline{\Omega} \mid \exists t \in (0,T) \text{ s.t. } \overline{X}(t,0,x) \in \mathbb{R}^2 \setminus \overline{\Omega} \right\} = \overline{\Omega}$$

Main result

Theorem (M. Badra, S.E., S. Guerrero 2014)

Under the above assumptions, there exists $\varepsilon > 0$ such that for all $(\rho_0, \mathbf{u}_0) \in L^\infty(\Omega) \times \mathbf{H}^1_0(\Omega)$ with $\operatorname{div}(\mathbf{u}_0) = 0$ satisfying

$$\|\rho_0\|_{L^\infty(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{H}^1_0(\Omega)} \leq \varepsilon,$$

there exists a trajectory

$$(\sigma, \mathbf{y}) \in L^\infty(\Omega_T) \times H^1(0, T; \mathbf{L}^2(\Omega)) \cap L^2(0, T; \mathbf{H}^2(\Omega))$$

satisfying the control requirement

$$(\sigma, \mathbf{y})(T) = (\overline{\sigma}, \overline{y})(T).$$

Related references

• On non-homogeneous incompressible Navier-Stokes equations:
⇝ Fernandez-Cara 2012: some optimal control problems.

• Local controllability for homogeneous incompressible Navier-Stokes equations:
⇝ Imanuvilov 2001, Fernandez-Cara Guerrero Imanuvilov Puel 2004, González-Burgos Guerrero Puel 2009, Imanuvilov Puel Yamamoto 2009, ...
Based on Carleman estimates for parabolic equations, see Fursikov, Imanuvilov 1996, and for elliptic equations to handle the pressure term, see Imanuvilov Puel 2003.

Related references (2)

• Controllability results for models involving both transport and parabolic effects:

- Thermoelasticity, see Albano Tataru 2001.
- Viscoelasticity, see Martin Rosier Rouchon 2013, Chaves-Silva Rosier Zuazua 2013,
- 1d Compressible Navier-Stokes equations
  $\rightsquigarrow$ cf the previous part, S.E. Glass Guerrero Puel 2012.

But here, we also have to handle the divergence free condition.

Strategy

Schauder fixed point argument

Setting $\rho \stackrel{\text{def}}{:=} \sigma - \overline{\sigma}$, $\mathbf{u} \stackrel{\text{def}}{:=} \mathbf{y} - \overline{\mathbf{y}}$, we have to construct solutions of

$$
\begin{array}{rcll}
\partial_t \rho + (\overline{\mathbf{y}} + \mathbf{u}) \cdot \nabla \rho & = & -\mathbf{u} \cdot \nabla \overline{\sigma} & \text{in } \Omega_T, \\
\overline{\sigma} \partial_t \mathbf{u} + \overline{\sigma}(\overline{\mathbf{y}} \cdot \nabla)\mathbf{u} + \overline{\sigma}(\mathbf{u} \cdot \nabla)\overline{\mathbf{y}} - \nu \Delta \mathbf{u} + \nabla p & = & \mathbf{f}(\rho, \mathbf{u}) & \text{in } \Omega_T, \\
\operatorname{div} \mathbf{u} & = & 0 & \text{in } \Omega_T, \\
(\rho(0), \mathbf{u}(0)) & = & (\rho_0, \mathbf{u}_0) & \text{in } \Omega, \\
(\rho(T), \mathbf{u}(T)) & = & (0, \mathbf{0}) & \text{in } \Omega, \\
\mathbf{u} & = & 0 & \text{on } (0, T) \times \Gamma_0.
\end{array}
$$

where

$$
\mathbf{f}(\rho, \mathbf{u}) \stackrel{\text{def}}{:=} -\rho(\partial_t \overline{\mathbf{y}} + (\overline{\mathbf{y}} \cdot \nabla)\overline{\mathbf{y}}) - \rho(\partial_t \mathbf{u} + ((\overline{\mathbf{y}} + \mathbf{u}) \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\overline{\mathbf{y}}) - \overline{\sigma}(\mathbf{u} \cdot \nabla)\mathbf{u}.
$$

Strategy, Simplified form

Schauder fixed point argument

Setting $\rho \overset{\text{def}}{:=} \sigma - \overline{\sigma}$, $\mathbf{u} \overset{\text{def}}{:=} \mathbf{y} - \overline{\mathbf{y}}$, we have to construct solutions of

$$\begin{array}{rcll} \partial_t \rho + (\overline{\mathbf{y}} + \widehat{\mathbf{u}}) \cdot \nabla \rho & = & -\mathbf{u} \cdot \nabla \overline{\sigma} & \text{in } \Omega_T, \\ \overline{\sigma} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p & = & \mathbf{f}(\rho, \mathbf{u}) & \text{in } \Omega_T, \\ \operatorname{div} \mathbf{u} & = & 0 & \text{in } \Omega_T, \\ (\rho(0), \mathbf{u}(0)) & = & (\rho_0, \mathbf{u}_0) & \text{in } \Omega, \\ (\rho(T), \mathbf{u}(T)) & = & (0, \mathbf{0}) & \text{in } \Omega, \\ \mathbf{u} & = & 0 & \text{on } (0, T) \times \Gamma_0. \end{array}$$

where $\mathbf{f}(\rho, \mathbf{u}) \overset{\text{def}}{:=} -\rho(\partial_t \overline{\mathbf{y}} + (\overline{\mathbf{y}} \cdot \nabla)\overline{\mathbf{y}})$.

$\rightsquigarrow$ Weak coupling $\rho \sim \mathbf{u}$

Strategy, decoupling into two control problems

The transport problem

Given $\widehat{\mathbf{u}}$, solve

$$\begin{aligned} \partial_t \rho + (\overline{\mathbf{y}} + \widehat{\mathbf{u}}) \cdot \nabla \rho &= -\widehat{\mathbf{u}} \cdot \nabla \overline{\sigma} && \text{in } \Omega_T, \\ \rho(0) = \rho_0, \quad \rho(T) &= 0 && \text{in } \Omega, \end{aligned}$$

The Stokes problem

Given $\rho, \widehat{\mathbf{u}}$, solve

$$\begin{aligned} \overline{\sigma} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}(\rho, \widehat{\mathbf{u}}) && \text{in } \Omega_T, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega_T, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}(T) &= \mathbf{0} && \text{in } \Omega, \\ \text{with } \mathbf{u} &= \mathbf{0} && \text{on } (0, T) \times \Gamma_0. \end{aligned}$$

Difficulties

In order to apply Schauder's fixed point theorem, two issues:

- Getting estimates for both problems in compatible spaces.
- Compactness properties.

Remark

Our strategy is close to the one of the previous section for the local exact controllability around constant trajectory of compressible Navier-Stokes equations.

The Stokes controlled problem

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$ satisfying $\Omega \subset \mathcal{O}$, $\partial\mathcal{O}$ is of class C^2, $\partial\mathcal{O} \cap \partial\Omega \supset \Gamma_0$, and consider the control problem in $\mathcal{O}_T = (0,T) \times \mathcal{O}$: Given $\mathbf{u}_0$, $\mathbf{f}$, find $\mathbf{h}$ such that

$$\begin{cases} \overline{\sigma}\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \nabla p & = & \mathbf{f} 1_{\Omega} + \mathbf{h} 1_{\mathcal{O}\setminus\overline{\Omega}} & \text{in } \mathcal{O}_T, \\ \operatorname{div} \mathbf{u} & = & 0 & \text{in } \mathcal{O}_T, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Gamma_T \stackrel{\text{def}}{:=} (0,T) \times \partial\mathcal{O}, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}(T) & = & \mathbf{0} & \text{in } \mathcal{O}. \end{cases}$$

Carleman weights functions

$$\tilde\psi \stackrel{\text{def}}{:=} \tilde\psi(t,x) \text{ s.t. } \begin{cases} \forall (t,x) \in \overline{\mathcal{O}_T},\ \tilde\psi(t,x) \in [0,1], \\ \forall (t,x) \in \Gamma_T,\ \partial_{\mathbf{n}} \tilde\psi(t,x) \leq 0, \\ \forall t \in [0,T],\ \tilde\psi(t)_{|\partial\mathcal{O}} \text{ is constant}, \\ \forall t \in [0,T],\ \inf_{\mathcal{O}} \tilde\psi(t,\cdot) = \tilde\psi(t)_{|\partial\mathcal{O}}. \\ \inf_{[0,T]\times(\mathcal{O}\setminus\overline{\Omega})} \{|\nabla\tilde\psi|\} \geq \alpha > 0. \end{cases}$$

For $m \geq 1$, and $\theta = \theta(t)$ a weight in time,

$$\psi(t,x) \stackrel{\text{def}}{:=} \tilde\psi(t,x) + 6m,$$

$$\varphi(t,x) \stackrel{\text{def}}{:=} \theta(t) \left(\lambda e^{6\lambda(m+1)} - \exp(\lambda\psi(t,x)) \right),$$

$$\xi(t,x) = \theta(t) \exp(\lambda\psi(t,x)).$$

The function θ

$$\theta_{m,\mu} \stackrel{\text{def}}{:=} \theta_{m,\mu}(t) \text{ s.t. } \begin{cases} \forall t \in [0, T_0],\ \theta_{m,\mu}(t) = 1 + \left(1 - \dfrac{t}{T_0}\right)^{\mu}, \\ \forall t \in [T_0, T - 2T_1],\ \theta_{m,\mu}(t) = 1, \\ \forall t \in [T - T_1, T),\ \theta_{m,\mu}(t) = \dfrac{1}{(T-t)^m}, \\ \theta_{m,\mu} \text{ is increasing on } [T - 2T_1, T - T_1], \\ \theta_{m,\mu} \in C^2([0, T)). \end{cases}$$

with $\mu = s\lambda^2 e^{\lambda(6m-4)}$.

Remarks

- Does not blow up as $t \to 0$.
- Strongly convexified close to $t = 0$.

Result

Theorem

Let $m \geq 5$. $\exists C > 0$, s.t. $\forall s \geq s_0$ and $\forall \lambda \geq \lambda_0$, if $\mathbf{f} \in \mathbf{L}^2(\mathcal{O}_T)$ satisfies

$$\iint_{\mathcal{O}_T} \xi^{-4} |\mathbf{f}|^2 e^{2s\varphi} < \infty,$$

there exists a control function $\mathbf{h} \in \mathbf{L}^2(\mathcal{O}_T)$ supported in $(0, T) \times (\mathcal{O} \setminus \overline{\Omega})$ and $\mathbf{u} \in \mathbf{L}^2(\mathcal{O}_T)$ solving the control problem and satisfying the estimate:

$$\|e^{(3/4)s\varphi}\mathbf{u}\|^2_{L^2(\mathbf{H}^2)\cap H^1(\mathbf{L}^2)} + s^{1/2} \iint_{\mathcal{O}_T} \xi^{-18/5} |\mathbf{u}|^2 e^{2s\varphi}$$
$$\leq C \iint_{\mathcal{O}_T} \xi^{-4} |\mathbf{f}|^2 e^{2s\varphi} + C \|e^{(5/4)s\varphi(0,\cdot)}\mathbf{u}_0\|^2_{\mathbf{H}^1_0(\mathcal{O})}.$$

On the proof

Steps of the proof:
Step 1: Carleman estimates for the heat equation with non-homogeneous boundary condition and $H^{-1}(\mathcal{O}_T)$ source terms:

- Carleman estimates for the heat equation with homogeneous boundary condition;
- Energy estimates on controlled trajectories of the heat equation;
- Duality;

$\rightsquigarrow$ In the spirit of Imanuvilov Yamamoto 2003.

On the proof

Step 2: Carleman estimates on the adjoint Stokes equations

$$\begin{cases} -\partial_t(\overline{\sigma}\mathbf{v}) - \nu\Delta\mathbf{v} + \nabla p = \mathbf{g} & \text{in } \mathcal{O}_T, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \mathcal{O}_T, \\ \mathbf{v} = 0 & \text{on } \Gamma_T, \end{cases}$$

- Apply H^{-1} Carleman estimates to $w = \operatorname{rot}\mathbf{v}$ satisfying

$$-\overline{\sigma}\partial_t w - \nu\Delta w = \operatorname{rot}\mathbf{g} + \partial_t\mathbf{v}\cdot\nabla^{\perp}\overline{\sigma} \quad \text{in } \mathcal{O}_T,$$

- Use Carleman estimates for the Laplace equation [Imanuvilov Puel 2003] and the stream function ζ ($\mathbf{v} = \nabla^{\perp}\zeta$) given by $\Delta\zeta(t,\cdot) = w(t,\cdot)$ on $\mathcal{O}$, $\zeta(t,\cdot) =$ constant on each connected component of $\partial\mathcal{O}$ to get rid of the boundary term.

Controlling the density

Recall the goal

The transport problem

Given $\widehat{\mathbf{u}}$, solve

$$\begin{aligned} \partial_t \rho + (\overline{\mathbf{y}} + \widehat{\mathbf{u}}) \cdot \nabla \rho &= -\widehat{\mathbf{u}} \cdot \nabla \overline{\sigma} \quad \text{in } \Omega_T, \\ \rho(0) = \rho_0, \quad \rho(T) &= 0 \quad \text{in } \Omega. \end{aligned}$$

$\rightsquigarrow$ As before, the natural idea is to construct forward and backward solutions and glue them.
Use

$$\left\{ x \in \overline{\Omega} \mid \exists t \in (0, T) \text{ s.t. } \overline{X}(t, 0, x) \in \mathbb{R}^2 \backslash \overline{\Omega} \right\} = \overline{\Omega}$$

Estimates

Do not forget that we need to get estimates in suitably weighted Sobolev spaces to run the fixed point argument.

Estimates on ρ

Let $\rho_0 \in L^\infty(\Omega)$. If

$$\partial_t \tilde{\psi} + \bar{\mathbf{y}} \cdot \nabla \tilde{\psi} = 0 \quad \text{in } \Omega_T,$$

we are able to derive $L^2(\Omega_T)$ and $L^\infty(\Omega_T)$ estimates, e.g.

$$\left\| \xi^{-2} e^{s\varphi} \rho \right\|_{L^2(\Omega_T)} \leq C \left(\left\| \xi^{-2} e^{s\varphi} \widehat{\mathbf{u}} \right\|_{\mathbf{L}^2(\Omega_T)} + e^{3s\varphi(0)/2} \left\| \rho_0 \right\|_{L^2(\Omega)} \right).$$

Putting together the two control problems

Existence of the weight function $\tilde{\psi}$.
$\tilde{\psi}$ should be $C^2([0, T] \times \overline{\mathcal{O}})$ and satisfy

$$\begin{cases} \forall (t, x) \in \Gamma_T,\ \partial_{\mathbf{n}} \tilde{\psi}(t, x) \leq 0, \\ \forall t \in [0, T],\ \tilde{\psi}(t)_{|\partial\mathcal{O}} \text{ is constant}, \\ \inf_{[0,T]\times(\mathcal{O}\setminus\overline{\Omega})} \{|\nabla \tilde{\psi}|\} \geq \alpha > 0, \\ \partial_t \tilde{\psi} + \overline{\mathbf{y}} \cdot \nabla \tilde{\psi} = 0 \quad \text{ in } \Omega_T. \end{cases}$$

We introduce an extension $\overline{\mathbf{y}}_e$ of $\overline{\mathbf{y}}$ such that

$$\overline{\mathbf{y}}_e = \overline{\mathbf{y}} \text{ in } (0, T) \times \Omega, \quad \overline{\mathbf{y}}_e \in \mathbf{C}^2(\overline{\mathcal{O}_T}),$$
$$\forall (t, x) \in (0, T) \times \partial\mathcal{O},\ \overline{\mathbf{y}}_e \cdot \mathbf{n} \geq \gamma_0 > 0,$$
$$\overline{\mathbf{y}}_e \text{ is “small” outside } \Omega.$$

Constructing the weight function

We then solve

$$\begin{cases} \partial_t \widehat{\psi} + \overline{\mathbf{y}}_e \cdot \nabla \widehat{\psi} = 0 & \text{in } \mathcal{O}_T, \\ \widehat{\psi}(t,x) = t - T & \text{on } \Gamma_T, \\ \widehat{\psi}(T) = \widehat{\psi}_T & \text{in } \mathcal{O}. \end{cases}$$

for $\widehat{\psi}_T$ satisfying

- $\widehat{\psi}_T$ is a non-negative $C^2(\overline{\mathcal{O}})$ function;
- The critical points of $\widehat{\psi}_T$ are outside Ω;
- Compatibility conditions on $\partial\mathcal{O}$:

$$\begin{cases} \widehat{\psi}_T(x) = 0 \text{ on } \partial\mathcal{O}, \\ \overline{\mathbf{y}}_e(T,x) \cdot \nabla \widehat{\psi}_T(x) = -1 \text{ on } \partial\mathcal{O}, \\ \partial_t \overline{\mathbf{y}}_e(T,x) \cdot \nabla \widehat{\psi}_T(x) - (\overline{\mathbf{y}}_e(T,x) \cdot \nabla)^2 \widehat{\psi}_T(x) = 0 \text{ on } \partial\mathcal{O}. \end{cases}$$

- $\inf_{\mathcal{O}} \widehat{\psi}_T = (\widehat{\psi}_T)_{|\partial\mathcal{O}} = 0.$

For concluding with Schauder's fixed point theorem, we introduce the map:

$$\mathscr{F} : \widehat{\mathbf{u}} \mapsto \mathbf{u},$$

1/ $\mathscr{F}$ maps a convex set into itself: Cf estimates.

$$\begin{aligned} \mathbf{X}_{s,\lambda}^R = \{\mathbf{u} \in & L^2(0,T;\mathbf{L}^2(\Omega)), \quad \text{with } \operatorname{div}(\mathbf{u}) = 0, \\ & s^{1/4}\xi^{-9/5}e^{s\varphi}\mathbf{u} \in L^2(0,T;\mathbf{L}^2(\Omega)), \\ & e^{3s\varphi/4}\mathbf{u} \in L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega)) \\ & \text{with } \|\mathbf{u}\|_{\mathbf{X}_{s,\lambda}} \leq R\}, \end{aligned}$$

for s, λ large enough, R small enough, $\rho_0, \mathbf{u}_0$ small enough.
2/ $\mathbf{X}_{s,\lambda}^R$ is compact in $L^2(0,T;\mathbf{L}^2(\Omega))$.
3/ $\mathscr{F}$ is continuous with respect to the $L^2(\mathbf{L}^2)$ topology ?

Here we use a result of Boyer Fabrie 2007:

Theorem (Boyer Fabrie 2007)

Let $\rho_0 \in L^\infty(\Omega)$ and $\rho_{in} \in L^\infty((0,T) \times \partial\Omega)$. Let $\mathbf{v}_k \in L^1(0,T; \mathbf{W}^{1,1}(\Omega))$ such that $\operatorname{div} \mathbf{v}_k = 0$ and $\mathbf{v}_k \cdot \mathbf{n} \in L^\delta((0,T) \times \partial\Omega)$ for some $\delta > 1$. Then, if $(\mathbf{v}_k)$ converges to $\mathbf{v}$ in $L^1(0,T; \mathbf{L}^1(\Omega))$ with $\mathbf{v} \in L^1(0,T; \mathbf{W}^{1,1}(\Omega))$, the sequence of solutions ρ_k of

$$\left\{ \begin{array}{rcll} \partial_t \rho_k + \mathbf{v}_k \cdot \nabla \rho_k & = & 0 & \text{in } (0,T) \times \Omega, \\ \rho_k(0) & = & \rho_0 & \text{in } \Omega, \\ \rho_k(t,x) & = & \rho_{in} & \text{for } (t,x) \in (0,T) \times \partial\Omega, \\ & & & \text{with } \mathbf{v}_k(t,x) \cdot \mathbf{n}(x) > 0 \end{array} \right.$$

strongly converges to the solution ρ corresponding to $\mathbf{v}$ in all spaces $L^q((0,T) \times \Omega)$ for $q \in (1,\infty)$.

This shows the continuity of the map $\mathscr{F}$ for the $L^2(\mathbf{L}^2)$ topology.

Further comments:

- the proof simplifies if $\partial_t \overline{\mathbf{y}} + \overline{\mathbf{y}} \cdot \nabla \overline{\mathbf{y}} = 0$ or $\nabla \overline{\sigma} = 0$ as this kills the linear coupling terms.
- When controlling on the whole boundary, there is local exact controllability around $(\overline{\rho}, \overline{\mathbf{y}}) = (1, \mathbf{0})$ in any time $T > 0$ by using Coron's return method and considering the trajectory $(1, \eta(t)\mathbf{U})$, $\mathbf{U} \in \mathbb{R}^2$ large and $\eta(t)$ a smooth bump function.

Open problems:

- Dimension 3 ?
- Global controllability to trajectories ? ...

Thank you for your attention!

Ref I: Local exact controllability for the 1-D compressible Navier-Stokes equation,
S.E., O. Glass, S. Guerrero and J.-P. Puel, ARMA 2012.

Ref II: Local controllability to trajectories for non-homogeneous 2d incompressible Navier-Stokes equations
M. Badra, S.E., S. Guerrero, 2014.