On the controllability of one dimensional degenerate parabolic equations with first order terms

LUZ DE TERESA

Control of PDEs

April 2014

Let us fix $T > 0$, a non-empty open subset $\omega \subset (0,1)$, we consider the linear system

$$\begin{cases} y_t - (x^\alpha y_x)_x + b(x,t)y + x^{\beta/2}c(x,t)y_x = h1_\omega \text{ in } Q = (0,1)\times(0,T), \\ y(1,t) = 0 \\ \begin{cases} y(0,t) = 0 \text{ if } 0 \leq \alpha < 1, \\ (x^\alpha y_x)(0,t) = 0 \text{ if } \ 1 \leq \alpha < 2, \end{cases} \quad t \in (0,T), \\ y(x,0) = y_0(x) \text{ in } \ (0,1). \end{cases} \tag{1}$$

where $b, c \in L^\infty(Q_1)$, $y_0 \in L^2(0,1)$, $\alpha \in [0,2)$ and $\beta \geq \alpha$

Formulation of the problem

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where $b, c \in L^\infty(Q_1)$, $y_0 \in L^2(0,1)$, $\alpha \in [0,2)$ and $\beta \ge \alpha$

Formulation of the problem

The above problem is well-posed in appropriate weighted spaces. For $0 \leq \alpha < 1$, define

$$H^1_\alpha(0,1) := \begin{array}{l} \{u \in L^2(0,1) |\ u \text{ abs. cont. in } [0,1], \\ x^{\alpha/2} u_x \in L^2(0,1); u(0) = u(1) = 0\}, \end{array}$$

and the unbounded operator $A\colon D(A) \subset L^2(0,1) \to L^2(0,1)$ by

$$D(A) := \left\{u \in H^1_\alpha(0,1) \mid x^\alpha u_x \in H^1(0,1)\right\}.$$

$$\forall\, u \in D(A), \quad Au := (x^\alpha u_x)_x,$$

Formulation of the problem

For $1 \leq \alpha < 2$, we define

$$H^1_\alpha(0,1) := \begin{array}{l}\{u \in L^2(0,1) |\ u \text{ locally absolutely continuous in } (0,1], \\ x^{\alpha/2} u_x \in L^2(0,1) \ \text{ and } \ u(1) = 0\}.\end{array}$$

Then, the unbounded operator $A\colon D(A) \subset L^2(0,1) \to L^2(0,1)$ will be defined by

$$D(A) := \begin{array}{l}\{u \in L^2(0,1)\ u \text{ locally absolutely continuous in } (0,1], \\ \quad x^\alpha u \in H^1_0(0,1), x^\alpha u_x \in H^1(0,1) \text{ and } (x^\alpha u_x)(0) = 0\}.\end{array}$$

$$\forall\, u \in D(A), \quad Au := (x^\alpha u_x)_x,$$

Theorem

Let h be given in $L^2(Q)$. For all $y_0 \in L^2(0,1)$, (1) *has a unique solution*

$$y \in \mathcal{U} := \mathcal{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_\alpha(0,1)).$$

Moreover, if $y_0 \in D(A)$, then

$$y \in \mathcal{C}^0([0,T]; H^1_\alpha(0,1)) \cap L^2(0,T; D(A)) \cap H^1(0,T; L^2(0,1)).$$

Formulation of the problem

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\begin{cases} y(0,t) = 0 \text{ if } 0 \le \alpha < 1,\\ (x^\alpha y_x)(0,t) = 0 \text{ if } \ 1 \le \alpha < 2, \end{cases} \quad t \in (0,T),\\
y(x,0) = y_0(x) \text{ in } \ (0,1).
\end{cases} \tag{1}
$$

Control problem: Does there exists $h \in L^2(Q)$ such that $y(T) = 0$?

Main result

Theorem

Given $T > 0$ and $y_0 \in L^2(0,1)$, there exists $h \in L^2(Q)$ such that the solution y of (1) satisfies

$$y(T) = 0 \ \ \textit{in} \ \ [0,1].$$

Moreover, for some positive constant C that depends on T,

$$\int_0^T \int_\omega |h|^2 dxdt \leq C \int_0^1 y_0^2(x) dx.$$

Previous results:

For

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- Null controllability result is true for any $T > 0$. (Cannarsa-Martinez-Vancostenoble and Martinez-Vancostenoble).
- $\alpha \ge 2$ Not null controllability at any time $T > 0$, regional controllability (Martinez-Vancostenoble).
- More general degeneracies (Alabau-Cannarsa-Fragnelli)

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- $$y_t - (x^\alpha y_x)_x = 0$$
  $$y(t,0) = h(t)$$
  (Gueye)
- For $\Omega \subset \mathbb{R}^2$ $y_t - \text{div}(A(x)\nabla y) = h1_\omega$ similar results (Cannarsa-Martinez-Vancostenoble)
- For $\Omega = (-1,1)\times(0,1)$, Grushing type operators, $v_t - \partial_x^2 v - |x|^\alpha \partial_y^2 = h1_\omega$ similar results (Beauchard-Cannarsa-Guglielmi) ($T > T^*$ critical case.)
- $$y_t - (x^\alpha y_x)_x + b(x,t)y + x^{\beta/2}c(x,t)y_x = h1_\omega \text{ in } Q,$$
  Regional null controllability result is true for any $T > 0$. (Cannarsa-Fragnelli-Vancostenoble).
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Observability inequality

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$$\begin{cases} v_t + (x^\alpha v_x)_x - b(x,t)v + (x^{\beta/2}c(x,t)v)_x = 0 & \text{in } Q_1, \\ v(1,t) = 0 & \\ \begin{cases} v(0,t) = 0 & \text{if } 0 \le \alpha < 1, \\ (x^\alpha v_x)(0,t) = 0 & \text{if } 1 \le \alpha < 2, \end{cases} & t \in (0,T), \\ v(x,T) = v_T(x), & \text{in } (0,1). \end{cases} \quad (3)$$

with $v_T \in L^2(0,1)$. Null controllability of (1) is equivalent to the following result:

Lemma

Let $T > 0$ be given. Then there exists a positive constant C such that every solution v of (3) *satisfies*

$$\int_0^1 v^2(x,0)dx \le C \int_0^T \int_\omega v^2(x,t)dxdt. \quad (4)$$

Observability inequality

Key points:

Hardy and Carleman inequalities

Hardy inequality

Hardy-Poincaré type inequality:

$$\int_0^1 x^{\alpha-2}u^2 dx \leq \frac{4}{(1-\alpha)^2}\int_0^1 x^\alpha {u_x}^2 dx,$$

where u is a locally absolutely continuous function in $(0,1)$ and

$u(x) \rightarrow_{x\rightarrow 0^+} 0$, if $0 \leq \alpha < 1$;
$u(x) \rightarrow_{x\rightarrow 1^-} 0$ if $1 < \alpha < 2$
and, in both cases, $\int_0^1 x^\alpha {u_x}^2 dx < \infty$.

Carleman inequality

Consider system

$$\begin{cases} v_t + (x^\alpha v_x)_x = F_0 + (x^{\beta/2} F_1)_x & \text{in } Q, \\ v(1,t) = 0 \quad \text{and} \begin{cases} v(0,t) = 0 & \text{if } 0 \le \alpha < 1, \\ (x^\alpha v_x)(0,t) = 0 & \text{if } 1 \le \alpha < 2, \end{cases} & t \in (0,T), \\ v(x,T) = v_T(x), & \text{in } (0,1), \end{cases} \tag{3}$$

where $F_0, F_1 \in L^2(Q_1)$ and $v_T \in L^2(0,1)$.

Lemma (Carleman inequality)

Let $0 \leq \alpha < 2$ and $T > 0$ be given. Then there exists $\Phi(x,t) = \frac{\psi(x)}{(t(T-t))^4}$ and two positive constants C and s_0, such that every solution v of (3) *satisfies, for all $s \geq s_0$,*

$$\int_0^T\int_0^1 \left(s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2\right) e^{2s\Phi(x,t)} dxdt$$
$$\leq C\left(\int_0^T\int_\omega e^{2s\Phi(x,t)} v^2 dxdt + \int_0^T\int_0^1 (F_0^2 + s^2\theta^3 x^{\beta-\alpha} F_1^2) e^{2s\Phi(x,t)} dxdt\right).$$

Proof of the observability inequality

Apply the Carleman inequality with $F_0 = b(x,t)v$ and $F_1 = c(x,t)v$: we get

$$\int_0^T\int_0^1 \left(s\theta x^{\alpha} v_x^2 + s^3\theta^3 x^{2-\alpha} v^2\right) e^{2s\Phi(x,t)} dxdt$$
$$\leq C\left(\int_0^T\int_\omega e^{2s\Phi(x,t)} v^2 dxdt + \int_0^T\int_0^1 (b^2v^2 + s^2\theta^3 x^{\beta-\alpha} c^2 v^2) e^{2s\Phi(x,t)} dxdt\right).$$

Proof of the observability inequality

Apply Hardy inequality to $w = e^{s\Phi(x,t)}v$ and get

$$\int_0^T \int_0^1 b^2 v^2 e^{2s\Phi(x,t)} dxdt \leq \int_0^T \int_0^1 (s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2) e^{2s\Phi(x,t)} dxdt$$

Combined with the Carleman inequality we get:

$$(1-\frac{C}{s})\left[\int_0^T\int_0^1\left(s\theta x^\alpha v_x^2+s^3\theta^3x^{2-\alpha}v^2\right)e^{2s\Phi(x,t)}dxdt\right]$$
$$\leq C\int_0^T\int_\omega e^{2s\Phi(x,t)}v^2dxdt.$$

Proof of the observability inequality

Combined with the Carleman inequality we get:

$$(1-\frac{C}{s})\left[\int_0^T\int_0^1\left(s\theta x^\alpha v_x^2+s^3\theta^3x^{2-\alpha}v^2\right)e^{2s\Phi(x,t)}dxdt\right]$$
$$\leq C\int_0^T\int_\omega e^{2s\Phi(x,t)}v^2dxdt.$$

Energy estimates and Hardy inequality imply:

$$\begin{aligned}\int_0^1 v^2(x,0)dx &\leq C\int_{T/4}^{3T/4}\int_0^1 x^\alpha v_x^2\\ &\leq C\int_{T/4}^{3T/4}\int_0^1 s\theta e^{2s\Phi(x,t)}x^\alpha v_x^2\\ &\leq C\int_0^T\int_\omega e^{2s\Phi(x,t)}v^2dxdt.\end{aligned}$$

How to prove Carleman Inequality?

Steps

How to prove Carleman Inequality? First prove

$F_0 = F_1 = 0$

$$\int_0^T\int_0^1 \left(s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2\right) e^{2s\Phi(x,t)}dxdt \leq \; C\int_0^T\int_\omega e^{2s\Phi(x,t)}v^2 dxdt.$$

(Cannarsa-deT)

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First prove

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(Cannarsa-deT)

- Carleman inequality for non-degenerate parabolic equations.
- Carleman inequality for degenerate parabolic equation.
- Adequate election of parameters

Steps

Imanuvilov-Yamamoto technique used two times. That is prove two auxiliary control problems

First auxiliary control problem

First control result

Given $f \in L^2(Q_1)$, find $u \in L^2(Q)$ such that the solution $z \in L^2(0,T;H^1_\alpha(0,1))$ to

$$\begin{cases} z_t - (x^\alpha z_x)_x = s^3\theta^3 x^{2-\alpha} e^{2s\Phi(x,t)} f + u 1_\omega & \text{in } Q, \\ z(1,t) = 0 & \\ \begin{cases} z(0,t) = 0 & \text{if } 0 \le \alpha < 1, \\ (x^\alpha z_x)(0,t) = 0 & \text{if } 1 \le \alpha < 2, \end{cases} & t \in (0,T), \\ z(x,0) = 0, & \text{in } (0,1), \end{cases}$$

satisfies $z(x,T) = 0$ in $(0,1)$

First Carleman inequality allows to find such u with:

$$\int_0^T \int_0^1 e^{-2s\Phi(x,t)} z^2 dxdt + \int_0^T \int_\omega e^{-2s\Phi(x,t)} u^2 dxdt$$
$$\leq C \int_0^T \int_0^1 s^3 \theta^3 x^{2-\alpha} e^{2s\Phi(x,t)} f^2 dxdt.$$

First auxiliary control problem

Let v solve:

$$v_t + (x^\alpha v_x)_x = F_0 + (x^{\beta/2} F_1)_x \text{ in } \quad Q$$

Take the auxiliary control problem:

Find a control $\hat{u} \in L^2(Q_1)$ and state $\hat{z} \in L^2(0,T;H^1_\alpha(0,1))$ such that

$$\hat{z}_t - (x^\alpha \hat{z}_x)_x = s^3\theta^3 x^{2-\alpha} e^{2s\Phi(x,t)} v + \hat{u}1_\omega \text{ in } \quad Q$$

$$\hat{z}(x,0) = \hat{z}(x,T) = 0 \text{ in } \quad (0,1),$$

and

$$\int_0^T\int_0^1 e^{-2s\Phi(x,t)}\hat{z}^2 dxdt + \int_0^T\int_\omega e^{-2s\Phi(x,t)}\hat{u}^2 dxdt$$
$$\leq \; C \; \int_0^T\int_0^1 s^3\theta^3 x^{2-\alpha} e^{2s\Phi(x,t)} v^2 dxdt.$$

Second auxiliary control problem

Let v solve:

$$v_t + (x^\alpha v_x)_x = F_0 + (x^{\beta/2} F_1)_x \text{ in } \quad Q$$

Take the second auxiliary control problem:

$$\tilde{z}_t - (x^\alpha \tilde{z}_x)_x = s\theta(e^{2s\Phi(x,t)} x^{\alpha/2} v_x)_x + \tilde{u} 1_\omega \text{ in } \quad Q$$

$$\tilde{z}(x,0) = \tilde{z}(x,T) = 0$$

$$
\begin{aligned}
&\int_0^T\int_0^1 e^{-2s\Phi(x,t)}\tilde{z}^2 dxdt + \int_0^T\int_\omega e^{-2s\Phi(x,t)}\tilde{u}^2 dxdt \\
&\qquad + \int_0^T\int_0^1 s^{-2}\theta^{-2}x^\alpha e^{-2s\Phi(x,t)}\tilde{z}_x^2 dxdt \\
&\leq \ C \ \int_0^T\int_0^1 s\theta x^\alpha e^{2s\Phi(x,t)} v_x^2 dxdt
\end{aligned}
$$

First auxiliary control problem implies:

$$\int_0^T\int_0^1 s^3\theta^3 x^{2-\alpha} e^{2s\Phi(x,t)} v^2 dxdt \leq C\left(\int_0^T\int_0^1 e^{2s\Phi(x,t)} F_0^2 dxdt + \int_0^T\int_0^1 s^2\theta^3 x^{\beta-\alpha} e^{2s\Phi(x,t)} F_1^2 dxdt + \int_0^T\int_\omega e^{2s\Phi(x,t)} v^2 dxdt\right).$$

Second auxiliary control problem implies:

$$\int_0^T\int_0^1 s\theta x^{\alpha} e^{2s\Phi(x,t)} v_x^2 dxdt$$
$$\leq C\left(\int_0^T\int_0^1 e^{2s\Phi(x,t)}\left[F_0^2 + s^2\theta^3 x^{\beta-\alpha} F_1^2\right] dxdt + \int_0^T\int_{\omega} e^{2s\Phi(x,t)} v^2 dxdt\right),$$

We are done

$$\int_0^T\int_0^1 s^3\theta^3 x^{2-\alpha} e^{2s\Phi(x,t)} v^2 dxdt + \int_0^T\int_0^1 s\theta x^{\alpha} e^{2s\Phi(x,t)} v_x^2 dxdt$$
$$\leq C\left(\int_0^T\int_0^1 e^{2s\Phi(x,t)}\left[F_0^2 + s^2\theta^3 x^{\beta-\alpha} F_1^2\right] dxdt + \int_0^T\int_\omega e^{2s\Phi(x,t)} v^2 dxdt\right),$$

Further results

All the results are valid with more general degeneracies:

$$\begin{cases} y_t - (a(x)y_x)_x + b(x,t)y + \beta(x)c(x,t)y_x = h1_\omega \text{ in } Q \\ y(1,t) = 0 \\ \begin{cases} y(0,t) = 0 \text{ if wdp} \\ (ay_x)(0,t) = 0 \text{ if sdp} \end{cases} \quad t \in (0,T), \\ y(x,0) = y_0(x) \text{ in } \ (0,1). \end{cases}$$

with $\beta(x)/x \in L^\infty(0,1)$, $a \in C([0,1]) \cap C^1((0,1])$, $a(0) = 0$, $a > 0$ in $(0,1]$.

Further results

and in the semi linear case

$$\begin{cases} y_t - (a(x)y_x)_x + f(t,x,y,y_x) = h1_\omega \text{ in } Q \\ y(1,t) = 0 \\ \begin{cases} y(0,t) = 0 \text{ if wdp} \\ (ay_x)(0,t) = 0 \text{ if sdp} \end{cases} \quad t \in (0,T), \\ y(x,0) = y_0(x) \text{ in } \ (0,1). \end{cases}$$

$f(\cdot; s,p) = g(\cdot; s,p)s + G(\cdot; s,p)p$ with $|\frac{G(x,t;s,p)}{\beta(x)}| < C$

Systems

$$\begin{cases} y_t - (a_1(x)y_x)_x + b(x,t)y = h1_\omega \text{ in } Q \\ v_t - (a_2(x)v_x)_x + c(x,t)v + d(x,t)y = 0 \\ y(1,t) = v(1,t) = 0 \\ \begin{cases} y(0,t) = 0 \text{ if wdp} \\ v(0,t) = 0 \text{ if wdp} \\ (a_1 y_x)(0,t) = 0 \text{ if sdp} \\ (a_2 v_x)(0,t) = 0 \text{ if sdp} \end{cases} \quad t \in (0,T), \\ y(x,0) = y_0(x);\ v(x,0) = v_0(x) \text{ in } \ (0,1). \end{cases}$$

- $a_1(x) = a_2(x) = x^\alpha$ Cannarsa-deT.
- $a_1(x) \neq a_2(x)$ E. M. Ait Ben Hassi, F. Ammar-Khodja, A. Hajjaj and L. Maniar

Thank you!
Merci!
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