Insensitizing controls for the Boussinesq system with vanishing components

Conservatoire National des Arts et Métiers
Control of PDE's

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Outline

Insensitizing controls

- Ω bounded connected regular open subset of $\mathbb{R}^N$ ($N = 2$ or 3)
- $T > 0$
- $\omega \subset \Omega$ (control set), $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$

We consider the Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f + v\mathbb{1}_\omega + (0,0,\theta), \quad \nabla\cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = f_0 + v_0\mathbb{1}_\omega & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau\widehat{y}_0, \quad \theta(0) = \theta^0 + \tau\widehat{\theta}_0 & \text{in } \Omega. \end{cases}$$

where τ is a small constant and $\|\widehat{y}^0\|_{L^2(\Omega)^3} = \|\widehat{\theta}^0\|_{L^2(\Omega)} = 1$. Unknown.

Insensitizing control problem: To find controls v and v_0 in $L^2(\omega \times (0,T))$ such that the functional (Sentinel)

$$J_\tau(y,\theta) := \frac{1}{2} \iint\limits_{\mathcal{O}\times(0,T)} \left(|y|^2 + |\theta|^2 \right) \mathrm{d}x\,\mathrm{d}t, \ \mathcal{O} \subset \Omega \ \text{(Observation set)}$$

is not affected by the uncertainty of the initial data, that is,

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial \tau}\right|_{\tau=0} = 0 \quad \forall\, (\widehat{y}_0, \widehat{\theta}_0) \in L^2(\Omega)^4 \text{ s.t. } \|\widehat{y}_0\|_{L^2(\Omega)^3} = \|\widehat{\theta}_0\|_{L^2(\Omega)} = 1.$$

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Some previous works

- Heat equation: [Bodart, Fabre, 1995], [de Teresa, 2000], [Bodart, González-Burgos, Pérez-García, 2002]
- Hyperbolic equations: [Alabau-Boussouira, 13]
- Quasi-Geostrophic ocean model: [Fernández-Cara, García, Osses, 2005]
- Stokes: [Guerrero, 2007]
- Navier-Stokes: [Gueye, 2013]

A cascade system

The previous condition is equivalent to the following null controllability problem: To find a control v and v_0 such that $z(0)=0$ and $q(0)=0$, where

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + \nabla p_0 = f + v\,\mathbb{1}_\omega + (0,0,r), & \nabla\cdot w = 0 \quad \text{in } Q,\\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + \nabla p_1 = w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z = 0 \quad \text{in } Q,\\ r_t - \Delta r + (w\cdot\nabla)r = f_0 + v_0\,\mathbb{1}_\omega & \text{in } Q,\\ -q_t - \Delta q - (w\cdot\nabla)q = z_3 + r\mathbb{1}_{\mathcal{O}} & \text{in } Q,\end{cases}$$

with boundary and initial conditions:

$$\begin{cases} w = z = 0, \quad r = q = 0 & \text{on } \Sigma,\\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & \text{in } \Omega.\end{cases}$$

We are interested in controls of the form

1. $v = (v_1, 0, 0)$, $v_0 \neq 0$
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Null controllability result

Assume:

- $y^0 = 0$, $\theta^0 = 0$
- $\mathcal{O} \cap \omega \neq \emptyset$
- $\|e^{K/t^{10}} f\|_{L^2(Q)^3} < +\infty$, $\|e^{K/t^{10}} f_0\|_{L^2(Q)} < +\infty$, some $K > 0$

Theorem (Guerrero, Gueye, C.)

There exists $\delta > 0$ such that if $\|e^{K/t^{10}}(f, f_0)\|_{L^2(Q)^4} < \delta$, there exist a controls (v, v_0) in $L^2(\omega \times (0, T))$ of the form $v = (v_1, 0, 0)$, $v_0 \neq 0$ such that $z(0) = 0$ and $q(0) = 0$.

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The linearized system around zero with source terms:

$$\begin{cases} w_t - \Delta w + \nabla p_0 = f^w + v\,\mathbb{1}_\omega + (0,0,r), & \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla p_1 = f^z + w\mathbb{1}_\mathcal{O}, & \nabla \cdot z = 0 & \text{in } Q, \\ r_t - \Delta r = f^r + v_0\,\mathbb{1}_\omega & & \text{in } Q, \\ -q_t - \Delta q = f^q + z_3 + r\mathbb{1}_\mathcal{O} & & \text{in } Q, \end{cases}$$

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$$v = (v_1, 0, 0), v_0 \neq 0 \quad \text{and} \quad v = (v_1, 0, v_3), v_0 = 0.$$

We prove an observability inequality for the adjoint system

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Adjoint system with source terms

Dual variables: $\varphi \leftrightarrow w, \quad \psi \leftrightarrow z, \quad \phi \leftrightarrow r, \quad \sigma \leftrightarrow q$

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi &= g^\varphi + \psi\,\mathbb{1}_{\mathcal{O}}, \qquad \nabla\cdot\varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi &= g^\psi + (0,0,\sigma), \quad \nabla\cdot\psi = 0 & \text{in } Q, \\ -\phi_t - \Delta\phi &= g^\phi + \varphi_3 + \sigma\,\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ \sigma_t - \Delta\sigma &= g^\sigma & \text{in } Q, \end{cases}$$

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Observability inequality

- For general controls $v = (v_1, v_2, v_3)$ and v_0:

$$\iint_Q \rho_1(t)(|\varphi|^2+|\psi|^2+|\phi|^2+|\sigma|^2) \leq C \iint_Q \rho_2(t)(|g^\varphi|^2+|g^\psi|^2+|g^\phi|^2+|g^\sigma|^2) + C \iint_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2+|\phi|^2)$$

$\rho_i(t) \sim \exp(-C/t^{10}(T-t)^{10})$

Using energy estimate, we can change to $\rho_i(t) \sim \exp(-C/t^{10})$

- By duality, if $\rho_1(t)^{-1/2}(f^w, f^z, f^\phi, f^\sigma)$ in $L^2(Q)$
  - $\rho_2(t)^{-1/2}(w, z, r, q)$ in $L^2(Q)$
  - $\rho_3(t)^{-1/2}(v, v_0)$ in $L^2(\omega\times(0,T))$
- For controls $v = (v_1, 0, 0)$ and v_0: only local terms φ_1 and ϕ
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Sketch of proof. Case: $v = (v_1, 0, 0)$ and v_0

- Carleman for φ_1 and φ_3.
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$$\Delta\psi_i = -\Delta\varphi_{i,t} - \Delta^2\varphi_i + \partial_i \nabla \cdot g^{\varphi} - \Delta g_i^{\varphi} \quad \text{in } \omega \cap \mathcal{O},\ i = 1, 3.$$

- Eliminate φ_3 using

$$\varphi_3 = -\phi_t - \Delta\phi - g^{\phi} - \sigma \quad \text{in } \omega \cap \mathcal{O}$$

- At this point we have local terms of φ_1, ϕ and global terms of σ. Carleman for σ, but cannot have a local term like σ.

$$(\partial_1^2 + \partial_2^2)\sigma = -(\partial_t^2 - \Delta^2)\Delta\varphi_3 + F(g^{\varphi}, g^{\psi}, g^{\sigma}) \quad \text{in } \omega \cap \mathcal{O}$$

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- At this point we have local terms of φ_1, ϕ and global terms of σ. Carleman for σ, but cannot have a local term like σ.

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- Carleman with a local term like $(\partial_1^2 + \partial_2^2)\sigma$

Sketch of proof. Case: $v = (v_1, 0, 0)$ and v_0

- Carleman for φ_1 and φ_3.
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$$\Delta\psi_i = -\Delta\varphi_{i,t} - \Delta^2\varphi_i + \partial_i \nabla \cdot g^\varphi - \Delta g_i^\varphi \quad \text{in } \omega \cap \mathcal{O},\ i = 1, 3.$$

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- Cannot add a local term of ϕ. No way to eliminate with coupling. Instead, we use energy estimates with weights like $\rho(t) = \exp(-C/t^{10})$:

$$\begin{cases} -(\rho\phi)_t - \Delta(\rho\phi) = \rho g^\phi + \rho\varphi_3 + \rho\sigma \mathbb{1}_{\mathcal{O}} - \rho'(t)\phi \\ (\rho\phi)_{|\Sigma} = 0, \quad (\rho\phi)(0) = 0, \quad \boxed{(\rho\phi)(T) = 0} \end{cases}$$

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Perspectives

- Our method limits the quantity of vanishing components to two. Also, we need to have v_3 or v_0
- What about three vanishing components, e.g., $v = (0, 0, 0)$ and v_0?
  One possibility: use the Return method.
  See P. Lissy's presentation in a few minutes.
- Boussinesq system: No control on the heat equation.

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v \mathbb{1}_\omega + (0,0,\theta), \quad \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = 0 & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega. \end{cases}$$

Adjoint equation

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Some references

N. C. and S. Guerrero, Local null controllability of the N-dimensional Navier-Stokes system with $N-1$ scalar controls in an arbitrary control domain, *J. Math. Fluid Mech.*, **15** (2013), no. 1, 139–153.

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Thank you for your attention!