Inverse problem for the waves : stability and convergence matters.

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Outline

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An inverse problem for the waves

Consider the wave equation in a smooth bounded domain Ω :

$$\begin{cases} \partial_{tt} y - \Delta_x y + p(x) y = f, & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = g(t,x), & (t,x) \in (0,T) \times \partial\Omega \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), & x \in \Omega. \end{cases}$$

- Given data : Source terms f, g ; initial data : (y^0, y^1).
- Unknown : the potential $p = p(x)$.
- Additional information : $\partial_\nu y(t,x)$ on $(0,T) \times \partial\Omega$.

Goal : Find the potential p. [Non-linear inverse problem]

Is it possible to retrieve the potential $p = p(x),\ x \in \Omega$ *from measurement of the flux* $\partial_\nu y(t,x)$ *on* $(0,T) \times \partial\Omega$ *?*

- Several related questions
  - Uniqueness : Given two potentials $p_1 \neq p_2$, can we guarantee $\partial_\nu y[p_1] \neq \partial_\nu y[p_2]$?
  - Stability : Given two potentials p_1, p_2, if $\partial_\nu y[p_1] \simeq \partial_\nu y[p_2]$, can we guarantee that $p_1 \simeq p_2$?
  - Reconstruction : Given $\partial_\nu y[p]$, can we compute p ?

- Known results : Uniqueness (Klibanov '92) and stability (Yamamoto '99, Imanuvilov Yamamoto '01), using Carleman estimates.

- Main Open Problem : Reconstruction ; how to compute the potential from the boundary measurement ?

A natural idea for reconstruction

Given a continuous measurement $\mathscr{M}[p] = \partial_\nu y[p]|_{(0,T)\times\partial\Omega}$

- Discretize the wave equation

$$\begin{cases} \partial_{tt} y_h - \Delta_h y_h + p_h y_h = f_h \simeq f, \\ y_h|_{(0,T)\times\partial\Omega} = g_h \simeq g, \\ (y_h, \partial_t y_h)(t=0) = (y_h^0, y_h^1) \simeq (y^0, y^1). \end{cases}$$

- Solve the following discrete inverse problem : Find a potential p_h so that the corresponding discrete solution $y_h[p_h]$ approximates at best the measurement :

$$\partial_h y_h[p_h]|_{(0,T)\times\partial\Omega}(t,x) \simeq \mathscr{M}[p](t,x)$$

$$\text{i.e. } p_h = \text{Argmin}_{q_h} \|\partial_h y_h[q_h] - \mathscr{M}[p]\|_*$$

Question : Do we get $p_h \simeq p$?

First goal :

Analyze the convergence of the discrete inverse problems.

Remarks :

- Natural question for all inverse problems in infinite dimensions : Finding a source term, a conductivity...
- Depends *a priori* on the numerical scheme employed.

Main difficulty :

- Different dynamics for the wave equation and its discrete approximations, cf Ervedoza - Zuazua ’11 :
  $\rightsquigarrow$ Numerical artefacts : High-frequency spurious waves, generated by the schemes.

Wave propagation, continuous and discrete media

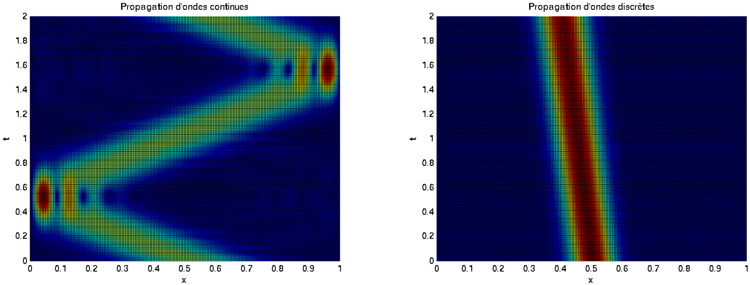


FIGURE: Wave propagation in continuous and discrete media

Continuous dynamics $\neq$ Discrete dynamics

Second goal :

Propose a new globally convergent algorithm for reconstruction.

Remarks :

- Reconstruction of the potential, with a single boundary measurement ;
- Using the observation $\mathscr{M}[p] = \partial_\nu y[p]$, a classical method for solving this inverse problem consists in minimizing

$$J(q) = \|\partial_t \left(\partial_\nu y[q] - \mathscr{M}[p]\right)\|^2_{L^2(\Gamma_0\times(0,T))}$$

$\rightsquigarrow$ not convex $\rightsquigarrow$ local minima ;

- Our algorithm will be based on Carleman estimate and the proof scheme of the stability result.

Outline

Inverse problem for the wave equation

- Generalities
- Reconstruction
- Goals

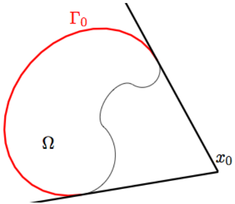
Lipschitz stability result for the continuous wave equation

Convergence of the discrete inverse problem

- Lax type Theorem
- Uniform stability estimates
- Higher dimension

Globally convergent reconstruction algorithm

Assumptions



Let $x_0 \in \mathbb{R}^N \setminus \Omega$ and let Γ_0 and T satisfy

$$\{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0 \quad ; \quad T > \sup_{x\in\Omega}\{|x - x_0|\}.$$

Let the potential p, the initial data y^0 and the solution $y[p]$ s.t.

$$\|p\|_{L^\infty(\Omega)} \leq m \qquad \inf\{|y^0(x)|, x \in \Omega\} \geq \gamma > 0$$

$$y[p] \in H^1(0,T; L^\infty(\Omega))$$

Then, one can prove uniqueness and local Lipschitz stability of the inverse problem for the continuous wave equation.

Theorem (Yamamoto '99, revisited LB '01)

Assume the above geometric constraints on (Γ_0, T, Ω).

Let $m > 0$ and $p \in L^\infty_{\leq m}(\Omega)$. Then $\exists C > 0$ depending only on

$$\inf_\Omega \{|y^0(x)|\} \ (\neq 0) \quad \text{and} \quad \|y[p]\|_{H^1(0,T;L^\infty(\Omega))},$$

such that for all $q \in L^\infty_{\leq m}(\Omega)$,

$$\frac{1}{C}\|p-q\|_{L^2(\Omega)} \leq \|\partial_\nu y[p] - \partial_\nu y[q]\|_{H^1((0,T);L^2(\Gamma_0))} \leq C\|p-q\|_{L^2(\Omega)}.$$

Remark : On $\Omega = (0,1)$, $\|y[p]\|_{H^1(0,T;L^\infty)}$ can be uniformly bounded for $p \in L^\infty_{\leq m}$ by taking smooth data...

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Carleman Estimate (Imanuvilov '02, Puel-LB '01)

Assuming $\Box = \partial_{tt} - \Delta_x$ and

$$\{x \in \partial\Omega,\ (x - x_0)\cdot \nu(x) > 0\} \subset \Gamma_0,$$

$\exists s_0 > 0$, $\lambda > 0$ and $M = M(s_0, \lambda, T, \beta, x_0) > 0$ such that :

$$s \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}(|\partial_t w|^2 + |\nabla w|^2)\, dxdt + s^3 \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|w|^2\, dxdt$$
$$\leq M \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\Box w|^2\, dxdt + M s \int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi}\, |\partial_\nu w|^2\, d\sigma dt$$

for all $s > s_0$ and $w \in L^2(-T, T; H_0^1(\Omega))$ satisfying

$$\begin{cases} \Box w \in L^2(\Omega \times (-T,T)), \\ \partial_\nu w \in L^2(-T,T; L^2(\Gamma_0)), \\ w(x, \pm T) = \partial_t w(x, \pm T) = 0,\ \forall x \in \Omega. \end{cases}$$

$\rightsquigarrow$ but also Zhang, Klibanov,...

Weights of the Carleman estimate

For the wave operator (with or without potential q)

$$\Box = \partial_{tt} - \Delta_x,$$

we let $\lambda > 0$ and $\beta \in (0,1)$ and define the weights ψ and φ by

$$\begin{aligned} \psi(x,t) &= |x - x_0|^2 - \beta t^2 + C_0, \\ \varphi(x,t) &= e^{\lambda \psi(x,t)} \end{aligned}$$

where $C_0 > 0$ is such that $\psi \geq 1$ on $\Omega \times [0,T]$.

The proof starts with setting $v = e^{s\varphi} w$ and defining

$Pv = e^{s\varphi} \Box (e^{-s\varphi} v) = P_1 v + P_2 v$ ...

Idea of the proof of Lipschitz stability

$$\begin{cases} \partial_{tt}y - \Delta_x y + p(x)y = f, & (0,T)\times\Omega, \\ y = g, & (0,T)\times\partial\Omega \\ (y(0),\partial_t y(0)) = (y^0,y^1), & \Omega. \end{cases}$$

Let $z = y[p] - y[q]$. Then z solves

$$\begin{cases} \partial_{tt}z - \Delta_x z + q(x)z = (q-p)y[p], & (t,x)\in(0,T)\times\Omega, \\ z(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega \\ (z(0,x),\partial_t z(0,x)) = (0,0), & x\in\Omega. \end{cases}$$

Set $Z = \partial_t z$ and extend the equation in negative time :

$$\begin{cases} \partial_{tt}Z - \Delta_x Z + q(x)Z = (q-p)\partial_t y[p], & (t,x)\in(-T,T)\times\Omega, \\ Z(t,x) = 0, & (t,x)\in(-T,T)\times\partial\Omega \\ (Z(0,x),\partial_t Z(0,x)) = (0,(q-p)y^0), & x\in\Omega. \end{cases}$$

- Apply the Carleman estimate to $w = \eta(t)Z$, where η is a cut-off function vanishing close to $t = \pm T$.

$$\begin{cases} \partial_{tt} w - \Delta_x w + q(x) w = \eta(q-p)\partial_t y[p] + \eta' \partial_t Z + \eta'' Z, \\ w(t,x) = 0, \\ (w(0,x), \partial_t w(0,x)) = (0, (q-p)y^0). \end{cases}$$

$\rightsquigarrow$ creates a source term localized in Supp $\eta' \cup$ Supp η'',

i.e. localized close to $\pm T$.

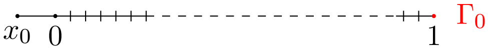
- But assuming $T > \sup_{x\in\Omega}\{|x - x_0|\}$,

$$\sup_{x\in\Omega} \psi(\pm T, x) = \sup\{|x-x_0|^2\} - \beta T^2 + C_0 \leq C_0 \leq \inf_{x\in\Omega} \psi(0,x).$$

- We conclude by an energy estimate.

Outline

Discretized setting



Consider the 1D wave equation observed at $x = 1$.

Discretization using the finite-difference method : $N + 1 = 1/h$,

$$(\Delta_h y_h)_j = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}, \quad \forall j \in \{1, \cdots, N\}$$

and consider

$$\begin{cases} \partial_{tt} y_j - (\Delta_h y_h)_j + p_j y_j = f_j, \\ y_0(t) = y_{N+1}(t) = 0, \\ (y_h, \partial_t y_h)(t = 0) = (y_h^0, y_h^1). \end{cases}$$

The observation is then denoted by $(\partial_h^- y)_{N+1}$.

Lax theorem type argument

A two steps proof of convergence

- Consistency : For any potential q, one can find discrete potentials q_h so that $q_h \underset{h\to 0}{\longrightarrow} q$ in $L^2(\Omega)$ and

$$\partial_t(\partial_h^- y_h[q_h])_{N+1} \underset{h\to 0}{\longrightarrow} \partial_{tx} y[q](t,1) \text{ in } L^2(0,T).$$

- Uniform Stability : There exists a constant C independent of $h > 0$ such that for all p_h, q_h,

$$\left\|p_h - q_h\right\|_{L^2(\Omega)} \leq C \left\|\partial_t(\partial_h^- y_h[p_h])_{N+1} - \partial_t(\partial_h^- y_h[q_h])_{N+1}\right\|_{L^2(0,T)} .$$

$\rightsquigarrow$ We will focus on the uniform stability estimates.

Discrete Carleman estimate for the waves

Assuming $q_h \in L^\infty_{\leq m}$, $L_h = \partial_{tt} - \Delta_h + q_h$, there exists $s_0 > 0$, $\lambda > 0$, $\varepsilon > 0$ and a constant $M = M(s_0, \lambda, \varepsilon, T, \beta, x_0, m) > 0$ such that for all $h \in (0,1)$, $s \in (s_0, \varepsilon/h)$

$$s \int_{-T}^{T} \int_{[0,1)} e^{2s\varphi}(|\partial_t w_h|^2 + |\partial_h^+ w_h|^2) dt + s^3 \int_{-T}^{T} \int_{(0,1)} e^{2s\varphi} |w_h|^2 dt$$
$$\leq M \int_{-T}^{T} \int_{(0,1)} e^{2s\varphi} |L_h w_h|^2 \, dt + M s \int_{-T}^{T} e^{2s\varphi(t,1)} \left|(\partial_h^- w_h)_{N+1}\right|^2 \, dt$$
$$+ M s \int_{-T}^{T} \int_{[0,1)} e^{2s\varphi} |h \partial_h^+ \partial_t w_h|^2 \, dt$$

for all w_h satisfying $w_{0,h}(t) = w_{N+1,h}(t) = 0$ on $(-T,T)$ and $w_h(\pm T) = \partial_t w_h(\pm T) = 0$.

Remarks

- Similarities with the continuous Carleman estimate : the weight function φ and the powers of s are the same.
- The range of s is limited to $s \leq \varepsilon / h$, expected in view of Boyer - Hubert - Le Rousseau '09,'10.
- On the extra term in $s\, e^{2s\varphi} |h \partial_h^+ \partial_t w_h|^2$
  - Needed ! Otherwise, this would imply observability for the discrete waves, uniformly with respect to $h > 0$, see Zuazua '05. Sharp scale.
  - Concentrated at high-frequencies

Consequence

Instead of getting the stability estimate

$$\|p_h - q_h\|_{L^2(\Omega)} \leq C \left\| \partial_t(\partial_h^- y_h[p_h])_{N+1} - \partial_t(\partial_h^- y_h[q_h])_{N+1} \right\|_{L^2(0,T)}$$

we get the following one :

$$\begin{aligned} &\|p_h - q_h\|_{L^2(\Omega)} \\ &\quad \leq C \left\| \partial_t(\partial_h^- y_h[p_h])_{N+1} - \partial_t(\partial_h^- y_h[q_h])_{N+1} \right\|_{L^2(0,T)} \\ &\qquad + C \left\| h\partial_h^+ \partial_{tt}(y_h[p_h] - y_h[q_h]) \right\|_{L^2((0,T);L^2([0,1)))}. \end{aligned}$$

This is enough to our purpose, since the added term $\rightharpoonup 0$.

1D case - Observation operator

We introduce, for $h > 0$, the operator

$$\begin{array}{rccl} \Theta_h: & L^\infty_{h,\leq m}(0,1) & \rightarrow & L^2(0,T) \times L^2(0,T; L^2_h(0,1)) \\ & p_h & \mapsto & \big(\partial_t(\partial_h^- y_h)_{N+1}[p_h], h\partial_h^+ \partial_{tt} y_h[p_h]\big), \end{array}$$

and its continuous analogous

$$\begin{array}{rccl} \Theta_0: & L^\infty_{\leq m}(0,1) & \rightarrow & L^2(0,T) \times L^2(0,T; L^2(0,1)) \\ & p & \mapsto & \big(\partial_t \partial_x y[p](\cdot,1), 0\big). \end{array}$$

$m > 0$ is fixed, and we know *a priori* that $p \in L^\infty_{\leq m}(0,1)$.

1D case - Convergence result

Theorem (LB & Ervedoza '13)

Under some regularity, mild convergence and positivity assumptions on the data, let $q_h \in L^\infty_{h,\leq m}(0,1)$ *be such that*

$$\Theta_h(q_h) \underset{h\to 0}{\longrightarrow} \Theta_0(p) \quad \textit{strongly in } L^2(0,T) \times L^2((0,T)\times(0,1)).$$

Then

$$q_h \underset{h\to 0}{\longrightarrow} p \quad \textit{in } L^2(0,1).$$

For instance, $q_h = \text{Argmin}_{p_h} \|\Theta_h(p_h) - \Theta_0(p)\|$.

$\rightsquigarrow$ The proof is based on the following consistency result.

1D case - Consistency results

Theorem (LB & Ervedoza '13)

Under the same assumptions, for any potential $p \in L^\infty_{\leq m}(0,1)$, *there exists* $p_h \in L^\infty_{h,\leq m}(0,1)$ *such that*

$$p_h \underset{h\to 0}{\longrightarrow} p \quad \text{in } L^2(0,1) \quad \text{and}$$

$$\Theta_h(p_h) \underset{h\to 0}{\longrightarrow} \Theta_0(p) \quad \text{in } L^2(0,T) \times L^2((0,T)\times(0,1)).$$

Moreover

$$\sup_{h\in(0,1)} \|y_h[p_h]\|_{H^1(0,T;L^\infty_h(0,1))} < \infty.$$

$\rightsquigarrow$ the convergence result is not empty !

Higher dimension

[LB, Ervedoza & Osses]

We may not have the geometric condition

$$\exists x_0 \in \mathbb{R}^N \setminus \Omega,\ s.t.\ \{x \in \partial\Omega,\ (x - x_0).\nu > 0\} \subset \Gamma.$$

- If it is satisfied, in a square, with uniform mesh $\simeq$ 1D case.
- If not, i.e. the measurement comes from an arbitrary part Γ of the boundary $\partial\Omega$:

$$\mathscr{M}[p] = \partial_\nu y[p]_{|(0,T)\times\Gamma},$$

  then we still have the continuous result of Bellassoued '04 to start with.
- We will only work in a square $\Omega = (0,1)^2$

Theorem (Bellassoued '04, revisited)

Assume that $\exists \Gamma_0 \subset \partial\Omega$ and $\mathcal{O} \subset \Omega$ such that :

- $\Gamma \subset \Gamma_0$ and Γ_0 satisfies the geometric condition ;
- $\mathcal{O}$ is a neighbourhood of Γ_0 in Ω ;
- q is known on $\partial\Omega$

Define

$\Lambda(Q,m) := \{q \in L^\infty(\Omega),$ s.t. $q|_{\mathcal{O}} = Q$ and $\|q\|_{L^\infty(\Omega)} \leq m\}.$

Assume further that $q^a \in \Lambda(Q,m)$ and

- $\inf_\Omega |y^0(x)| \geq \gamma > 0$;
- $\left\|y^0\right\|_{H^1(\Omega)} \leq K$; $\|y[q^a]\|_{H^1(0,T;L^\infty(\Omega))\cap W^{2,1}(0,T;L^2(\Omega))} \leq K$.

Let $M > 0$ and $\alpha > 0$. Then $\exists C > 0$ such that for $T > 0$ large enough, $\forall q^b \in \Lambda(Q, m)$ satisfying (more a priori knowledge)

$$q^a - q^b \in H_0^1(\Omega) \text{ and } \left\| q^a - q^b \right\|_{H_0^1(\Omega)} \leq M,$$

then we have

$$\left\| q^a - q^b \right\|_{L^2(\Omega)} \leq C \left[\log \left(2 + \frac{C}{\| \mathscr{M}[q^a] - \mathscr{M}[q^b] \|_{H^1(0,T;L^2(\Gamma))}} \right) \right]^{-\frac{1}{1+\alpha}}.$$

About the proof

- Main tools
  - Fourier-Bros-Iagolnitzer type transform
    Links wave equations to elliptic equations :
    Robbiano '91, '95, Lebeau Robbiano '97 ;
  - (Global) Carleman estimate on elliptic equations :
    Hörmander ;
  - Global Carleman estimate on wave equations ;
- Sketch of the proof
  - Logarithmic stability result in $\mathcal{O}$;
  - Lipschitz stability results under the geometric condition.

A glimpse of the use of the FBI transform

We set the FBI transform of y as $Y_\lambda(s) = \int_{\mathbb{R}} \rho_\lambda(t + \mathbf{i}s) y(t)\, dt$,
ρ_λ being an holomorphic approximation of the identity when $\lambda \to \infty$;

$$\Rightarrow \partial_s Y_\lambda(s) = -\mathbf{i} \int_{\mathbb{R}} \partial_t \left(\rho_\lambda(t + \mathbf{i}s)\right) y(t)\, dt = \int_{\mathbb{R}} \rho_\lambda(t + \mathbf{i}s) \mathbf{i}\partial_t y(t)\, dt.$$

The wave equation becomes an elliptic equation :

$$\partial_{tt} y - \Delta y = f \quad \rightsquigarrow \quad -\partial_{ss} w_\lambda - \Delta w_\lambda = g,$$

where $$w_\lambda(s, x) = \int_{\mathbb{R}} \rho_\lambda(t + \mathbf{i}s)(\eta(t) y(t, x))\, dt.$$

We use elliptic Carleman estimates for w_λ. The point is to be able to derive estimates on y from these estimates on w_λ. It will rely on the properties of the kernel ρ_λ...

2D case - discrete stability results

Theorem (LB, Ervedoza, Osses '13)

Similar setting as in the continuous case, with $\Omega = (0,1)^2$.

q_h known on $\partial\Omega$, and on $\mathcal{O}$... Let $M > 0$ and $\alpha > 0$. Then $\exists C > 0$ such that for $T > 0$ large enough, $\forall q_h^b \in \Lambda_h(Q_h, m)$ satisfying $q_h^a - q_h^b \in H_0^1(\Omega)$ and $\left\| q_h^a - q_h^b \right\|_{H^1_{0,h}(\Omega)} \leq M$, we have

$$\left\| q_h^a - q_h^b \right\|_{L^2_h(\Omega_h)} \leq C h^{1/(1+\alpha)}$$

$$+ C \left[\log \left(2 + \frac{C}{\left\| \mathscr{M}_h[q_h^a] - \mathscr{M}_h[q_h^b] \right\|_{H^1(0,T;L^2(\Gamma))}} \right) \right]^{-\frac{1}{1+\alpha}}$$

$$+ C h \sum_{k=1,2} \left\| \partial_{h,k}^+ \partial_{tt} y_h[q_h^a] - \partial_{h,k}^+ \partial_{tt} y_h[q_h^b] \right\|_{L^2(0,T;L^2_h(\Omega_{h,k}^-))}.$$

Remarks & consequences

- We use the same tools in the semi-discrete framework
  - FBI type transform, using the sophisticated kernel of Lebeau Robbiano '97, Phung '10
  - Carleman estimates on semi-discrete elliptic equations : Boyer Hubert Le Rousseau '10
  - Discrete 2D hyperbolic Carleman estimate
- We get almost identifiability result
- Allows to derive convergence results in 2D, with the same observation operator Θ_h.

Outline

Towards a (re)constructive approach

[LB, de Buhan & Ervedoza]

Remember that $Z = \partial_t \left(y[p] - y[q] \right)$ satisfies

$$\begin{cases} \partial_{tt} Z - \Delta_x Z + q(x) Z = (q-p)\partial_t y[p], & (t,x) \in (0,T) \times \Omega, \\ Z(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega \\ (Z(0,x), \partial_t Z(0,x)) = (0, (q-p)y^0), & x \in \Omega. \end{cases}$$

Main idea : the source term $(q-p)\partial_t y[p]$ is less relevant than the initial data $(q-p)y^0$, whereas

$$\partial_\nu Z = \partial_t \partial_\nu y[p] - \partial_t \partial_\nu y[q] \quad \text{on } (0,T) \times \Gamma_0 \quad \text{is known.}$$

$\rightsquigarrow$ Hence, we try to fit Z using this information, and apply the following new Carleman estimate.

A new Carleman estimate [LB, de Buhan, Ervedoza '13]

Assuming $q \in L^\infty_{\leq m}(\Omega)$, $L_q = \partial_{tt} - \Delta_x + q(x)$,

$$\{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0\ , \qquad \sup_{x\in\Omega} |x - x_0| < \beta T$$

$\exists s_0 > 0$, $\lambda > 0$ and $M = M(s_0, \lambda, T, \beta, x_0, m) > 0$ such that

$$s\int_0^T\!\!\int_\Omega e^{2s\varphi}\left(|\partial_t w|^2 + |\nabla w|^2 + s^2|w|^2\right)\,dxdt + s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t w(0)|^2\,dx$$

$$\leq\ M\int_0^T\!\!\int_\Omega e^{2s\varphi}|L_q w|^2\,dxdt + Ms\int_0^T\!\!\int_{\Gamma_0} e^{2s\varphi}\,|\partial_\nu w|^2\ d\sigma dt,$$

for all $s > s_0$ and $w \in L^2(-T,T;H^1_0(\Omega))$ satisfying

$$\begin{cases} L_q w \in L^2(\Omega\times(-T,T)) \\ \partial_\nu w \in L^2((0,T)\times\Gamma_0), \\ w(0,x) = 0,\ \forall x \in \Omega. \end{cases}$$

An approximation

$$L_q = \partial_{tt} - \Delta_x + q(x).$$

If we set (using a strictly convex quadratic functional)

$$\tilde{Z}[s,q] = \underset{z/z(0,\cdot)=0}{\text{Argmin}} \left[\int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \partial_\nu Z|^2 + \frac{1}{s} \int_0^T \int_\Omega e^{2s\varphi} |L_q z|^2 \right]$$

then $\tilde{Z}$ is “close to” Z since we can prove, using the Carleman estimate, that

$$\int_\Omega e^{2s\varphi(0,x)} |\partial_t \tilde{Z}(0,x) - \partial_t Z(0,x)|^2 \, dx \leq \frac{C}{\sqrt{s}} \|\partial_t y[p]\|^2_{L^2(0,T;L^\infty(\Omega))} \int_\Omega e^{2s\varphi(0,x)} |p-q|^2 \, dx.$$

But $\partial_t Z(0,x) = (q-p)y^0(x)$. Hence, we set

$$q_1 = q - \frac{\partial_t \tilde{Z}[s,q](0,x)}{y^0(x)} \Leftrightarrow \partial_t \tilde{Z}[s,q](0,x) = (q-q_1)y^0(x),$$

and we have

$$\int_\Omega e^{2s\varphi(0,x)}|p-q_1|^2\,dx \leq \frac{C}{\sqrt{s}}\frac{\|\partial_t y[p]\|^2_{L^2(L^\infty)}}{\inf_\Omega\{|y^0(x)|\}^2}\int_\Omega e^{2s\varphi(0,x)}|p-q|^2\,dx.$$

For s large enough, this yields a convergent iterative algorithm where at each step, one solves a minimization problem for a strictly convex functional.

Algorithm

Initialization : $q^0 = 0$.

Iteration : Given q^k,

1 - Compute $w[q^k]$ the solution of

$$\begin{cases} \partial_t^2 w - \Delta w + q^k w = g, & \text{in } \Omega \times (0,T), \\ w = 0, & \text{on } \partial\Omega \times (0,T), \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega, \end{cases}$$

and set $\mu^k = \partial_t \left(\partial_\nu w[q^k] - \partial_\nu w[p] \right)$ on $\Gamma_0 \times (0,T)$.

2 - Introduce the functional

$$J_0^k(z) = \int_0^T \int_\Omega e^{2s\varphi} |L_{q^k} z|^2 + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2,$$

on the space $\mathcal{T}^k = \{ z \in L^2(0,T; H_0^1(\Omega)), z(0) = 0,$

$$L_{q^k} z \in L^2(\Omega \times (0,T)), \partial_\nu z \in L^2(\Gamma_0 \times (0,T)) \}.$$

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Theorem

Assume the **geometric and time conditions***. Then, for all $s > 0$ and $k \in \mathbb{N}$, the functional J_0^k is continuous, strictly convex and coercive on $\mathcal{T}^k$ endowed with a suitable weighted norm.*

3 - Let Z^k be the unique minimizer of the functional J_0^k, and then set

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t Z^k(0)}{w_0},$$

where w_0 is the initial condition of (1).

4 - Finally, set

$$q^{k+1} = T_m(\tilde{q}^{k+1}), \quad \text{where } T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \text{sign}(q)m, & \text{if } |q| \geq m. \end{cases}$$

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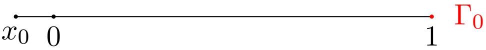
Numerical Simulations

- $\Omega = [0,1]$, $x_0 = -0.1$, $\Gamma_0 = \{x = 1\}$, $g = 0$, $\beta = 0.99$, $T = 1.5$, $\lambda = 0.1$, $s = 1$;



- Initial data $w_1 = 0$, $w_0(x) = 2 + \sin(\pi x)$ and first guess $q_0(x) = 0$;
- Finite differences in space $h = 0.02$, explicit Euler scheme in time $\tau = 0.01$;
- Additional noise on the observation data :

$$\mu = (1 + \alpha \, \text{Normal}(0, 0.5))\, \mu, \quad \alpha \geq 0;$$

- Minimization of J_0 by a conjugate gradient.

As seen in the discrete Carleman estimates, a regularization term must be added to make them uniform wrt the discretization parameter h. Thus, the '$J^k_{0,h}(z_h)$ formula' has an extra term

$$s \int_0^T \int_0^1 e^{2s\varphi} |h\partial_h^+ \partial_t z_h|^2 \, dt.$$

In less than $k = 20$ iterations,

- Without noise, for $p = \sin(2\pi x)$, one has

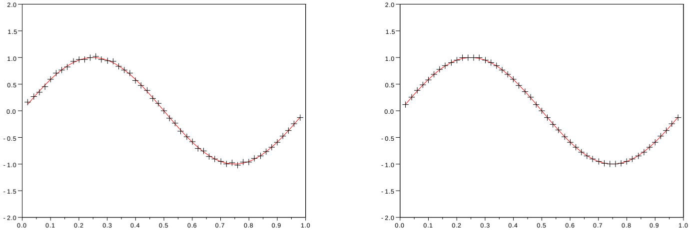
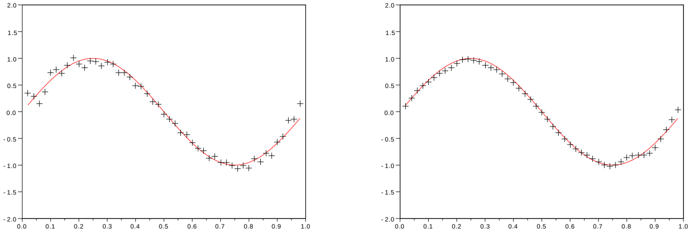
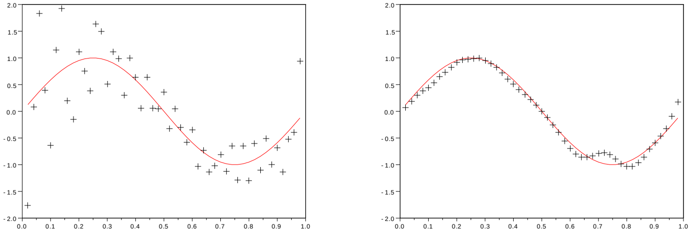


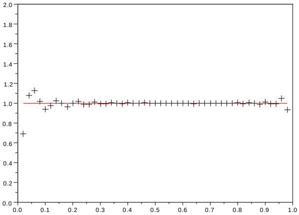
FIGURE: Without (left) and with (right) regularization term.

- Noise parameter $\alpha = 2\%$

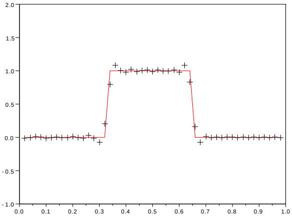


- Noise parameter $\alpha = 10\%$

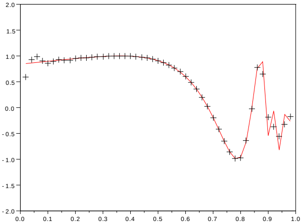




$p(x) = 1$



$p(x) = 0$ or 1



$p(x) = \sin(1 - 1/x)$

$\alpha = 0$; with regularization term.

Thank you for your attention !

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