On quantitative compactness estimates for hyperbolic conservation laws and Hamilton-Jacobi equations

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(Joint research project with O. Glass, Khai T. Nguyen and P. Cannarsa)

Control of PDEs

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Outline

1. Introduction
   - General setting
   - Kolmogorov entropy measure of compactness
2. Compactness estimates for conservation laws
   - Upper estimate for conservation laws
   - Lower estimates for conservation laws
3. Compactness estimates for HJ equations
   - Upper estimates
   - Lower estimates

Outline

Scalar conservation laws

Consider a scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \qquad x \in \mathbb{R}, \tag{1}$$

where

- $u = u(t, x)$ is the state variable
- the flux $f = f(u)$ is (uniformly) strictly convex

$$f''(u) \geq c > 0 \quad \forall u \in \mathbb{R}.$$

Entropy weak solutions

Distributional weak solution of (1)

$$\int\int [u\partial_t \varphi + f(u)\partial_x \varphi] = 0 \quad \forall \varphi \in C_c^1(]0,+\infty[\times\mathbb{R}). \tag{2}$$

Lax stability condition for admissibility

$$u(t,x-) \geq u(t,x+) \quad \text{for a.e } t > 0, \quad \forall x \in \mathbb{R}. \tag{3}$$

u is an entropy admissible weak solution of (1) if u satisfies (2) and (3).

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Semigroup $(S_t)_{t\geq 0}$

The scalar conservation law (1) generates an L^1-contractive semigroup

$$S_t : L^1(\mathbb{R}) \to L^1(\mathbb{R}), \qquad t > 0,$$

which associates to every given initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the unique entropy admissible weak solution $u(t,x)$ of (1), with initial datum $u(0,\cdot) = u_0$

$$S_t(u_0) \doteq u(t,\cdot).$$

Lax P. D., CPAM (1954)

$S_t : L^1(\mathbb{R}) \longrightarrow L^1_{\text{loc}}(\mathbb{R})$ is a compact operator for every $t > 0$

- Lax's question: is it possible to give a quantitive estimate of the compactness of S_t ?
- What about the semigroup S_t generated by a system of conservation laws?

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Kolmogorov ε-entropy

Let (X, d) be a metric space, K a totally bounded subset of X.

For $\varepsilon > 0$, let $N_\varepsilon(K)$ be the minimal number of sets in a cover of K by subsets of X having diameter $\leq 2\varepsilon$.

Definition

The ε-entropy of K is defined as

$$H_\varepsilon(K \mid X) \doteq \log_2 N_\varepsilon(K)$$

Problem: provide estimates on Kolmogorov's ε-entropy of

$$S_T(\mathcal{C}), \qquad \mathcal{C}: \text{ bounded set of initial data}$$

for semigoup map S_T generated by:

- a conservation law or a system of conservation laws (w.r.t. L^1-topology)
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Applications: rely on Kolmogorov's ε-entropy to:

- provide estimates on the accuracy and resolution of numerical methods
- analyze computational complexity (derive minimum number of needed operations to compute solutions with an error $< \varepsilon$)
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Upper compactness estimates $(f''(u) \geq c > 0)$

Given any $L, m, M > 0$, consider

$$\mathcal{C}_{[L,m,M]} := \left\{ u_0 \in L^1(\mathbb{R}) \mid Supp(u_0) \subset [-L, L], \|u_0\|_{L_1} \leq m, \|u_0\|_{L^\infty} \leq M \right\}.$$

Goal: Provide an estimate on

$$H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]}) | L^1(\mathbb{R})).$$

Upper estimate of ε-entropy:

C. De Lellis and F. Golse, CPAM (2005)

For any $T > 0$, and for $\varepsilon > 0$ sufficiently small one has

$$H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]}) \mid L^1(\mathbb{R})) \leq \frac{1}{\varepsilon} \cdot \frac{24 L(T)^2}{cT},$$

with

$$L(T) = L + 2 \sup_{|z| \leq M} |f''(z)| \sqrt{\frac{2mT}{c}}.$$

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Sketch of the proof

Key role: $f'' \geq c > 0 \implies$ Oleinik estimate

$$u(t,y) - u(t,x) \leq \frac{y-x}{c\,t}, \qquad x < y \qquad (u(t,x) \doteq S_t(u_0)(x)).$$

Equivalently, the function $x \mapsto \frac{x}{ct} - u(t,x)$ is increasing.

Thus, $\frac{\cdot}{ct} - S_T(\mathcal{C}_{[L,m,M]})$ is a set of bounded, compactly supported and increasing functions.

Lemma

For $L > 0$ and $V > 0$ set

$$\mathcal{I}_{L,V} = \{w : [0,L] \to [0,V] \mid w \text{ is nondecreasing}\}.$$

Then, for $0 < \varepsilon \leq \frac{LV}{6}$, the following holds:

$$H_\varepsilon(\mathcal{I}_{L,V} \mid L^1([0,L])) \leq \frac{1}{\varepsilon} \cdot 4LV.$$

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By the upper and lower bounds, we conclude

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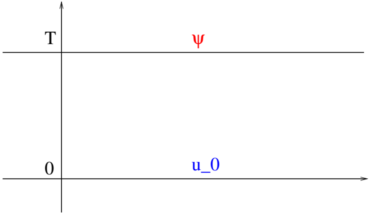
   Introduce a suitable parametrized class $\mathcal{F}$ of piecewise affine functions and show that

$$\mathcal{F} \subset S_T(\mathcal{C}_{[L,m,M]})$$

For $\Psi \in \mathcal{F}$, find $u_0 \in \mathcal{C}_{[L,m,M]}$

s.t. $S_T(u_0) = \Psi$



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   Provide an optimal (w.r.t. the parameters) estimate of the maximum number of functions $C_{2\varepsilon}$ in $\mathcal{F}$ that can be contained in a ball of radius 2ε (w.r.t. L^1 distance)

$$\Longrightarrow \qquad N_\varepsilon(\mathcal{F} \mid L^1(\mathbb{R})) \geq \frac{\text{Card}\{\mathcal{F}\}}{C_{2\varepsilon}} \approx \exp\left(\frac{1}{\varepsilon}\right)$$

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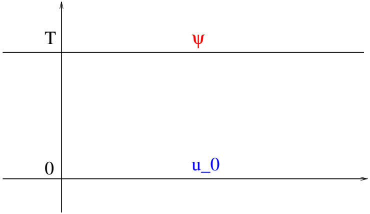

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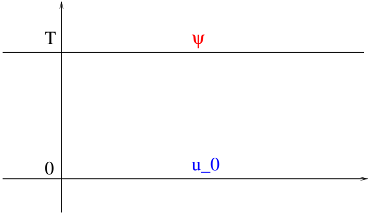

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$$\Longrightarrow \qquad N_\varepsilon(\mathcal{F} \mid L^1(\mathbb{R})) \geq \frac{\text{Card}\{\mathcal{F}\}}{C_{2\varepsilon}} \approx \exp\left(\frac{1}{\varepsilon}\right)$$

Reachability of piecewise C^1 functions

Consider the sets

$$\mathcal{C}_{[L,m,M]} \doteq \{u_0 \in L^1(\mathbb{R}) \mid Supp(u_0) \subset [-L, L], \|u_0\|_{L_1} \leq m, \|u_0\|_{L^\infty} \leq M\}$$

$$\mathcal{A}_{[L,M,b]} \doteq \left\{\psi : \mathbb{R} \to [-M, M] \;\middle|\; \mathrm{Supp}(\psi) \subset [-L, L], \psi \text{ is piecewise } C^1, |\psi'| \leq b\right\}$$

Proposition 1.

Given any $L, M, m, T > 0$, for h sufficiently small, one has

$$\mathcal{A}_{[L_T,h,b_T]} \subset S_T(\mathcal{C}_{[L,m,M]}),$$

with $L_T \doteq L - \alpha Th$, $b_T \doteq \frac{1}{\alpha T}$, $\alpha \doteq sup_{|u|\leq h}|f''(u)|$.

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GOAL: given $\psi \in \mathcal{A}_{[L_T,h,b_T]}$, find $u_0 \in \mathcal{C}_{[L,m,M]}$ s.t. $S_T(u_0) = \psi$

Backward construction: reversing the direction of time

$$w_0(x) \doteq \psi(-x), \quad w(t,x) \doteq S_t(w_0)(x).$$

Set

$$u(t,x) \doteq w(T-t,-x), \quad (t,x) \in [0,T] \times \mathbb{R}.$$

Observe that

$$u(T,\cdot) = \psi.$$

Moreover

$$|\psi'| \le b_T \quad \Longrightarrow \quad |u_x(t,x)| \ \text{ bounded on } [0,T] \times \mathbb{R}.$$

Therefore

- u is a classical solution (entropy admissible) of cons law $\Longrightarrow u(t,x) = S_t u_0(x)$ on $[0,T] \times \mathbb{R}$
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Piecewise affine functions in $\mathcal{A}_{L,M,b}$

Introduce a two-parameter class $\mathcal{F}$ of piecewise affine functions $\mathcal{F}_\iota \in \mathcal{A}_{L,M,b}$

Given $n \in \mathbb{N}$, $n > 1$ and $h > 0$, for every n-tuple $\iota = (\iota_i)_{i=0,1,\dots,n-1} \in \{0,1\}^n$, construct $\mathcal{F}_\iota$ as follows

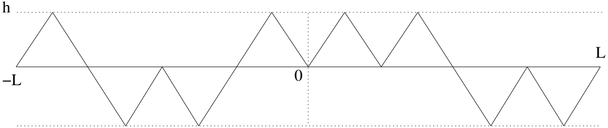

Figure : A function $\mathcal{F}_\iota$ for $\iota = (0,1,1,0,0,0,1,1)$

$$h \le M, \qquad \frac{nh}{L} \le b$$

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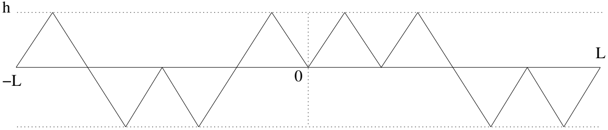

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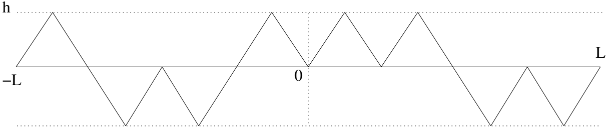

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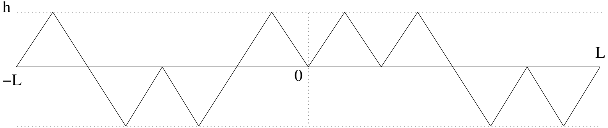

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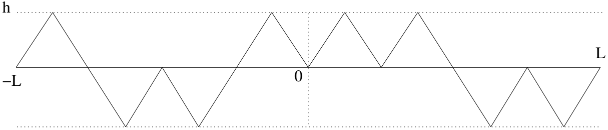

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Lower estimate for $H_\varepsilon(\mathcal{F} \mid L^1(\mathbb{R}))$

For any $\iota, \bar{\iota} \in \{0,1\}^n$, one has

$$\|\mathcal{F}_\iota - \mathcal{F}_{\bar{\iota}}\|_{L^1} \leq \frac{2hL}{n} d(\iota, \bar{\iota}).$$

where $d(\iota, \bar{\iota}) \doteq \text{Card}\,\{k \in \{1, \ldots, n\} \mid \iota_k \neq \bar{\iota}_k\}$ It follows that

$$\|\mathcal{F}_\iota - \mathcal{F}_{\bar{\iota}}\|_{L^1} \leq \varepsilon \quad \Longleftrightarrow \quad d(\iota, \bar{\iota}) \leq \frac{n\varepsilon}{2hL}.$$

Therefore, for any fixed $\bar{\iota} \in \{0,1\}^n$, let C_ε be the number of $\mathcal{F}_\iota$ such that $\|\mathcal{F}_\iota - \mathcal{F}_{\bar{\iota}}\|_{L^1} \leq \varepsilon$ (such a number is independent of $\bar{\iota}$).

We have

$$C_\varepsilon = \sum_{\ell=0}^{\lfloor \frac{n\varepsilon}{2hL} \rfloor} \binom{n}{\ell} = \frac{1}{2^n} \mathbb{P}\left(X_1 + \ldots + X_n \leq \left\lfloor \frac{n\varepsilon}{2hL} \right\rfloor \right)$$

where $X_1, \ldots, X_n$ are indep. random variables with Bernoulli distribution $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2}$.

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By Hoeffding's inequality, one obtains

$$C_\varepsilon \leq 2^n \exp\left(-\frac{n}{2}\left(1-\frac{\varepsilon}{Lh}\right)^2\right) \underset{\max\limits_{h,n}}{\leq} 2^n \exp\left(-\frac{1}{\varepsilon}\frac{4bL^2}{27}\right).$$

(taking $h = \frac{2bL}{n}$, $n = \frac{2bL^2}{3\varepsilon}$)

Notice: any element of an ε-cover of $\mathcal{F}$ contains at most $C_{2\varepsilon}$ functions of $\mathcal{F}$.

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$$C_\varepsilon \leq 2^n \exp\left(-\frac{n}{2}\left(1-\frac{\varepsilon}{Lh}\right)^2\right) \underset{\max\limits_{h,n}}{\leq} 2^n \exp\left(-\frac{1}{\varepsilon}\frac{4bL^2}{27}\right).$$

(taking $h = \frac{2bL}{n}$, $n = \frac{2bL^2}{3\varepsilon}$)

Notice: any element of an ε-cover of $\mathcal{F}$ contains at most $C_{2\varepsilon}$ functions of $\mathcal{F}$.

$$\mathrm{Card}\{\mathcal{F}\} = 2^n \quad \Longrightarrow \quad N_\varepsilon(\mathcal{F}) \doteq \left[\text{minimal} \ \# \ \text{ of sets in a } \varepsilon\text{-cover of } \mathcal{F}\right]$$

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$$N_\varepsilon(\mathcal{F}) \geq \frac{2^n}{C_{2\varepsilon}} \geq \exp\left(\frac{1}{\varepsilon} \cdot \frac{2bL^2}{27}\right)$$

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- $\mathcal{F} \subset \mathcal{A}_{[L_T,h,b_T]}$

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Lower compactness estimates for systems

Consider a strictly hyperbolic system of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \qquad x \in \mathbb{R}, \qquad u \in \Omega \subset \mathbb{R}^N$$

$(\lambda_1(u) < \cdots < \lambda_N(u)$ eigenvalues of $Df(u)$ with eigenvectors $r_i)$

F. A., O. Glass and K. T. Nguyen (2013)

Given any $L, m, M > 0$, consider

$$\mathcal{C}^{\delta_0}_{[L,m,M]} \doteq \Big\{ u_0 \in \mathbf{L}^1(\mathbb{R}) \mid Supp(u_0) \subset [-L, L], \|u_0\|_{\mathbf{L}^1} \le m, \|u_0\|_{\mathbf{L}^\infty} \le M, \\ \text{Tot.Var.}\{u_0\} < \delta_0 \Big\}, \qquad \delta_0 \ll 1.$$

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Outline of the proof

1. Let $s \mapsto R_i(s)$ denote the integral curve of the i-th eigenvector r_i, starting at the origin.
   Consider a family of profiles of i-simple waves $\{\phi_i^\iota\}_\iota$ defined as parametrizations
   $$s \mapsto \phi_i^\iota(s) \doteq R_i(\beta_\iota(s))$$
   of R_i through a suitable class of piecewise affine, compactly supported functions $\{\beta_\iota\}_\iota$.

2. Given any $L, m, M, T > 0$, for h sufficiently small, and for any profile of superposition $\phi^{\iota_1,\dots,\iota_N}$ of simple waves $\phi_1^{\iota_1}, \dots, \phi_N^{\iota_N}$, with $|\beta_\iota| \leq h$, $|\beta_\iota{}'| \leq b_T$, for all $\iota = \iota_1, \dots, \iota_N$, one has
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3. Consider a two-parameter class $\mathcal{F}$ of profiles of superposition $\phi^{\iota_1,\dots,\iota_N}$ of simple waves associated to a two-parameter class $\mathcal{B}$ of N-tuples of piecewise affine, compactly supported functions $(\beta_{\iota_1},\dots,\beta_{\iota_N})$ so that

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4. Observe that, setting

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Compactness estimates for Temple systems

Consider a strictly hyperbolic system of conservation laws of Temple class:

- endowed with a coordinates system $w = (w_1, \ldots, w_N)$ of Riemann invariants $w_i = w_i(u)$ associated to each characteristic field r_i;
- the level sets $\{u \in \Omega;\ w_i(u) = constant\}$ of every Riemann invariant are hyperplanes;

with all characteristic family genuinely nonlinear.

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From conservation laws to HJ equations

Given $u(t,x)$ entropy weak solution of the conservation law

$$\partial_t u + \partial_x f(u) = 0, \qquad x \in \mathbb{R},$$

the function

$$v(t,x) \doteq \int_{-\infty}^{x} u(t,z)dz$$

is a viscosity solution of the Hamilton-Jacobi equation

$$\partial_t v + f(\partial_x v) = 0, \qquad x \in \mathbb{R}.$$

Notice: quantitative compactness estimates for conservation laws w.r.t. L^1-topology plus Poincaré inequality $\Longrightarrow$ quantitative compactness estimates for H-J w.r.t. $W^{1,1}$-topology in 1-dim case.

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General setting

Consider a HJ equation

$$\begin{cases} u_t(t,x) + H(\nabla_x u(t,x)) = 0, \\ u(0,\cdot) = u_0(\cdot) \in \mathrm{Lip}(\mathbb{R}^n) \end{cases} \tag{4}$$

where $u : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ and $H \in C^2(\mathbb{R}^n)$ satisfies:

(UC) uniform convexity: $D^2H(p) \geq \alpha \cdot \mathbb{I}_n$, $\alpha > 0$.

For every $u_0(\cdot) \in \mathrm{Lip}(\mathbb{R}^n)$, (4) admits a unique viscosity solution u given by Hopf's formula

$$u(t,x) = \min_{y\in\mathbb{R}^n} \left\{ t \cdot L\left(\frac{x-y}{t}\right) + u_0(y) \right\}.$$

($L(q) = \max_{p\in\mathbb{R}^n}\{< p,q > -H(p)\}$ Legendre transform of H)

One has that

- $u(t,\cdot)$ is Lipschitz,
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Hopf-Lax semigroup

The H-J eqn generates a semigroup

$$S_t : \mathrm{Lip}(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n), \qquad t > 0,$$

which associates to every given initial data $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$, the unique viscosity solution $u(t,x)$ of (1), with initial datum $u(0,\cdot) = u_0$

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Provide upper and lower estimates on

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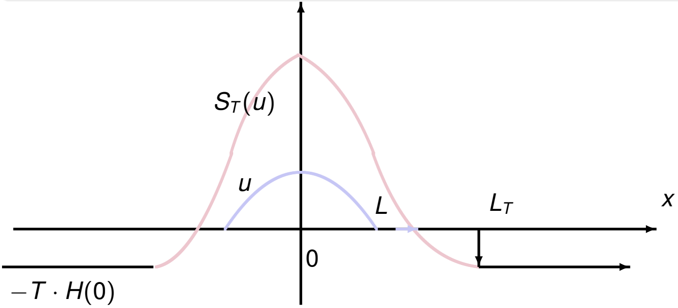

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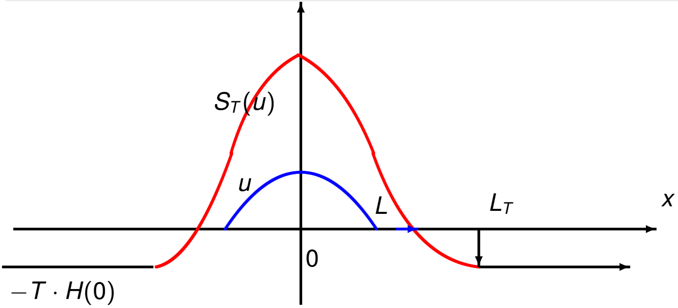

Outline

1. Introduction
   - General setting
   - Kolmogorov entropy measure of compactness
2. Compactness estimates for conservation laws
   - Upper estimate for conservation laws
   - Lower estimates for conservation laws
3. Compactness estimates for HJ equations
   - Upper estimates
   - Lower estimates

Upper estimates

F.A., P. Cannarsa, K.T. Nguyen (2013)

Let H satisfy **(UC)** and $\nabla H(0) = 0$. For any $T > 0$, and for $\varepsilon > 0$ sufficiently small, one has

$$H_\varepsilon(S_T(\mathcal{C}_{[L,M]}) + T \cdot H(0) \mid W^{1,1}(\mathbb{R}^n)) \leq \Gamma^+ \cdot \frac{1}{\varepsilon^n}$$

for some constant $\Gamma^+ > 0$ depending on L, M.

Main steps of the proof:

1. $S_T(\mathcal{C}_{[L,M]}) + T \cdot H(0) \subset \mathcal{SC}_{[\frac{1}{\alpha T}, L_T, M]}$ where $L_T \doteq L + \sup_{|p| \leq M} |\nabla H(p)|$ and

$$\mathcal{SC}_{[K,L,M]} \doteq \{u \in \mathcal{C}_{[L,M]} \mid u \text{ semiconcave with constant } K\}$$

2. $u \in \mathcal{SC}_{[K,L,M]} \implies x \mapsto D^+u(x) - Kx$ is monotone decreas. multif. ($D^+u(x)$ superdifferential of u at x)

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Upper estimates . . . continued

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4. relying on 2.-3. derive an upper bound for the ε-entropy of semiconcave functions

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F.A., P. Cannarsa, K.T. Nguyen (2013)

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Therefore,

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Main steps toward lower estimates on $H_\varepsilon(S_T(\mathcal{C}_{[L,M]}) \mid \mathbb{W}^{1,1}_{\text{loc}}(\mathbb{R}^n))$

1. Controllability type result.
   Introduce a suitable parametrized class $\mathcal{U}$ of smooth functions defined as combinations of suitable bump functions, and show that any element of such a class, up to a translation by a fixed map ψ, can be obtained, at any given time T, as the value $u(T, \cdot)$ of a viscosity solution with initial data in $\mathcal{C}_{[L,M]}$.

2. Combinatorial computation.
   Provide an optimal (w.r.t. the parameters) estimate of the maximum number of functions in such a class $\mathcal{U}$ that can be contained in a ball of radius 2ε (w.r.t. $\mathbb{W}^{1,1}$ distance).

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Reachability of semiconcave functions

Proposition 1

Let H satisfy **(UC)** and $\nabla H(0) = 0$. Given any $K, L, M > 0$ and $T > 0$ such that

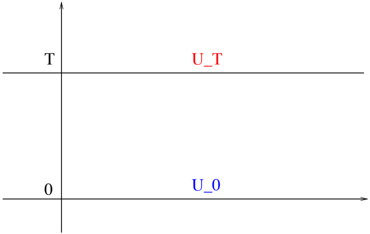
$$K \leq \frac{1}{4\|D^2 H(0)\| \cdot T}$$

for m sufficiently small one has

$$SC_{[K,L/2,m]} - T \cdot H(0) \subset S_T(C_{[L,M]}).$$

Goal: for $u_T \in SC_{[K,L/2,m]} - T \cdot H(0)$, we find $u_0 \in C_{[L,M]}$ such that

$$S_T(u_0) = u_T.$$



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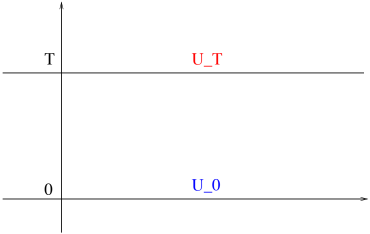

Backward construction

Reversing the direction of time

$$w_0(x) := u_T(-x), \quad w(t,x) = S_t(w_0)(x).$$

Set

$$u(t,x) = -w(T-t,-x), \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Observe that

- $u(T,\cdot) = u_T(\cdot)$,
- $u_0(\cdot) := u(0,\cdot) \in C_{[L/2,M]}$,
- For almost every $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$u_t(t,x) + H(\nabla_x u(t,x)) = 0.$$

$\Longrightarrow$ to prove $u_T \in S_T(C_{[L,M]})$, we need to show that $u(t,x)$ is a viscosity solution.

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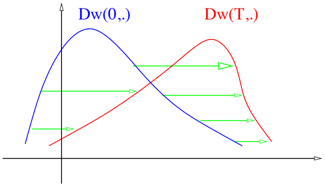
AIM: show that $w(t,x)$ is smooth in $]0,T[\times\mathbb{R}^n$.
Since $w(0,\cdot) = -u_T(-\cdot)$ is semiconvex with semiconvexity constant $-K \geq -\frac{1}{4\|D^2H(0)\|\cdot T}$, if $\mathrm{Lip}[w(0,\cdot)]$ is sufficiently small, one has

$$w(t,\cdot) \qquad \text{is semiconvex} \qquad \forall\, t \leq T$$



$\Longrightarrow$ $w(t,\cdot)$ is both semiconcave and semiconvex, hence $w(t) \in C^1$

Therefore, u is a classical solution of H-J equation in $[0,T]\times\mathbb{R}^n$ and hence it is a viscosity solution. It implies that

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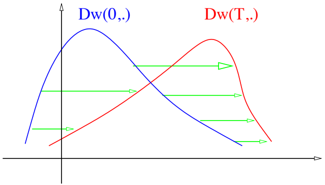

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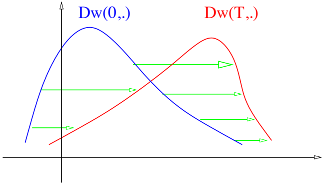

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Lower estimates of $\mathcal{H}_\varepsilon(SC_{[K,L,M]} \mid \mathbb{W}^{1,1}(\mathbb{R}^n))$

Proposition 2

Given $K, L, M > 0$, for $\varepsilon > 0$, it holds

$$\mathcal{H}_\varepsilon(SC_{[K,L,M]} \mid \mathbb{W}^{1,1}(\mathbb{R}^n)) \geq \Gamma^- \cdot \frac{1}{\varepsilon^n}.$$

Sketch of proof ($n = 2$):

Given $N \in \mathbb{Z}^+$, we divide $[0, L]^2$ into N^2 squares

$$[0, L]^2 = \bigcup_{\iota \in \{1,\dots,N\}^2} \square_\iota .$$

A bump function $b : \square \to \mathbb{R}$ such that

- $\mathrm{Lip}[b] \leq \frac{KL}{12N}$, $\|b\|_{\mathbb{W}^{1,1}} \leq \frac{C}{N^3}$
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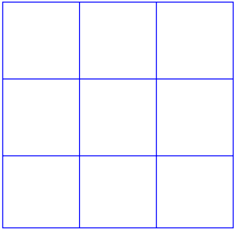

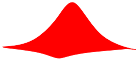

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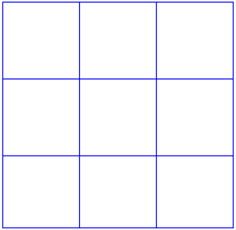

A class of smooth functions

Let

$$\Delta_N := \left\{ \delta = (\delta_\iota)_{\iota \in \{1,\ldots,N\}^2} \;\middle|\; \delta_\iota \in \{-1, 1\} \right\}$$

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$$\mathcal{U}_N := \left\{ u_\delta = \sum_{\iota \in \{1,\ldots,N\}^2} \delta_\iota \cdot b_n^\iota \;\middle|\; \delta \in \Delta_N \right\}.$$

For N sufficiently large, one has that

$$\mathcal{U}_N \subset SC_{[K,L,M]}.$$

On the other hand, by choosing $N \approx \frac{1}{\varepsilon}$ we have

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Merci de votre attention!

Viscosity solutions

A function $u \in \mathcal{C}([0,T] \times \mathbb{R}^n)$ is a viscosity solution of

$$u_t + H(t, x, \nabla u) = 0 \qquad \text{in }];[0,T] \times \mathbb{R}^n$$

if for every $(t,x) \in (0,T) \times \mathbb{R}^n$ and every $\phi \in \mathcal{C}^1((0,T) \times \mathbb{R}^n)$

- $u - \phi$ has a local maximum at $(t,x) \;\Rightarrow\; \phi_t(t,x) + H(t,x,\nabla\phi(t,x)) \leqslant 0$
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