Influence of the coupling on under-controlled cascade systems of PDE's

Fatiha Alabau-Boussouira

Université de Lorraine

CNAM, Conference Control of PDE's, march 31-april 4 2014

beamer-tu-lo

beamer-ur-logo

Outline

Sommaire

We consider

• under-controlled systems, that is: systems for which there are **less controls than equations (or unknowns)**,

• and moreover, which have a specific structure, here a cascade structure.

⇝ Our general purpose will be to study the influence of the coupling, in various aspects, on the controllability properties of these under-controlled cascade systems.

A model example is given by:

$$\begin{cases} y_{1,tt}(t,x) - \Delta y_1(t,x) + c(x) y_2(t,x) = 0\,, \quad (t,x) \in (0,T)\times\Omega\,, \\ y_{2,tt}(t,x) - \Delta y_2(t,x) = 0, (t,x) \in (0,T)\times\Omega \rightsquigarrow \text{decoupled from the 1st eq.} \\ y_1 = 0 \text{ in } (0,T)\times\Gamma\,, y_2 = b(x) v \text{ in } (0,T)\times\Gamma\,, \\ (y_i, y_{i,t})(0,.) = (y_i^0, y_i^1)\,, \quad \text{in } \Omega\,, \text{for } i = 1,2\,, \end{cases}$$

Here

- Ω is a bounded open set in $\mathbb{R}^d$ with a sufficiently smooth boundary Γ,
- the initial data (y_i^0, y_i^1), $i = 1,2$ are given in a suitable energy space,
- the functions c and b are respectively the coupling coefficient and the control coefficient, and can be resp. localized on some subdomains of Ω and Γ (or of Ω for internal control).
- v is the boundary control.

Similarly, one can also consider the case of a locally distributed control for cascade systems:

$$\begin{cases} y_{1,tt} - \Delta y_1 + c(x) y_2 = 0 \text{ in } (0,T)\times\Omega\,, \\ y_{2,tt} - \Delta y_2 = bv \text{ in } (0,T)\times\Omega\,, \rightsquigarrow \text{ decoupled from the first equation} \\ y_1 = y_2 = 0 \text{ in } (0,T)\times\Gamma\,, \\ (y_1, y_{1,t})(0,.) = (y_1^0, y_1^1)\,, (y_2, y_{2,t})(0,.) = (y_2^0, y_2^1) \text{ in } \Omega\,. \end{cases}$$

This problem can be reformulated as

$$Y'' + \mathcal{M}_2 Y = \mathcal{B}_2 v\,, (Y, Y')(0) = (y_1^0, y_2^0, y_1^1, y_2^1)\,,$$

where $Y = (y_1, y_2)^t$, $\mathcal{B}_2 v = (0, bv)^t$ and where the involved matrix operator $\mathcal{M}_2$ has the following upper triangular form

$$\mathcal{M}_2 = \begin{pmatrix} A & cI \\ 0 & A \end{pmatrix} \rightsquigarrow \text{this is the } \textit{cascade} \text{ structure,}$$

where I stands for the identity operator in $L^2(\Omega)$ and $A = -\Delta$ stands for the homogeneous Dirichlet Laplacian.

Similarly, one can also consider the case of a locally distributed control for cascade systems:

$$\begin{cases} y_{1,tt} - \Delta y_1 + c(x) y_2 = 0 \text{ in } (0,T)\times\Omega\,, \\ y_{2,tt} - \Delta y_2 = bv \text{ in } (0,T)\times\Omega\,, \rightsquigarrow \text{ decoupled from the first equation} \\ y_1 = y_2 = 0 \text{ in } (0,T)\times\Gamma\,, \\ (y_1, y_{1,t})(0,.) = (y_1^0, y_1^1)\,, (y_2, y_{2,t})(0,.) = (y_2^0, y_2^1) \text{ in } \Omega\,. \end{cases}$$

This problem can be reformulated as

$$Y'' + \mathcal{M}_2 Y = \mathcal{B}_2 v\,, (Y, Y')(0) = (y_1^0, y_2^0, y_1^1, y_2^1)\,,$$

where $Y = (y_1, y_2)^t$, $\mathcal{B}_2 v = (0, bv)^t$ and where the involved matrix operator $\mathcal{M}_2$ has the following upper triangular form

$$\mathcal{M}_2 = \begin{pmatrix} A & cI \\ 0 & A \end{pmatrix} \rightsquigarrow \text{this is the } \textit{cascade} \text{ structure,}$$

where I stands for the identity operator in $L^2(\Omega)$ and $A = -\Delta$ stands for the homogeneous Dirichlet Laplacian.

We can also consider other models, such as cascade systems of plates equations:

$$\begin{cases} y_{1,tt}(t,x) + \Delta^2 y_1(t,x) + c(x) y_2(t,x) = 0\,, \quad (t,x) \in (0,T) \times \Omega\,, \\ y_{2,tt}(t,x) + \Delta^2 y_2(t,x) = b(x) v(t,x)\,, \quad (t,x) \in (0,T) \times \Omega \\ y_1 = \frac{\partial y_1}{\partial \nu} = 0 \text{ in } (0,T) \times \Gamma\,, y_2 = \frac{\partial y_2}{\partial \nu} = 0\,, \text{ in } (0,T) \times \Gamma\,, \\ (y_i, y_{i,t})(0,.) = (y_i^0, y_i^1)\,, \quad \text{in } \Omega\,, \text{for } i = 1,2\,, \end{cases}$$

in the case of locally distributed control and

other type of coupled systems: with variable coefficients, beams...

for locally distributed as well as boundary control.

Both cascade systems of wave and plate equations . . . can be modeled by the following controlled abstract cascade system of two equations

$$Y'' + \mathcal{M}_2 Y = \mathcal{B}_2 \mathbf{v}\,,$$

with $Y = (y_1, y_2)^t$, $\mathcal{B}_2 \mathbf{v} = (0, B_2 v)^t$, and

$$\mathcal{M}_2 = \begin{pmatrix} A & C_{21} \\ 0 & A \end{pmatrix}\,,$$

where C_{21} is the coupling operator (a multiplication operator in our examples).

Here once again, only the last equation is controlled, and the purpose is to control the full system.

We can consider more generally bi-diagonal *n*-coupled controlled cascade system:

$$Y'' + \mathcal{M}_n Y = \mathcal{B}_n \mathbf{v}\,,$$

with $Y = (y_1, y_2, \dots y_n)^t$, $\mathcal{B}_n\mathbf{v} = (0, \dots, B_n v)^t$, and

$$\mathcal{M}_n = \begin{pmatrix} A & C_{21} & 0 & \dots & \\ 0 & A & C_{32} & 0 & \dots \\ \vdots & & & & \\ 0 & 0 & \dots & A & C_{nn-1} \\ 0 & 0 & \dots & 0 & A \end{pmatrix},,$$

where the C_{ii-1} are coupling operators.

Here once again, only the last equation is controlled, and the purpose is to control the full system.

Of course, one can consider:

• Full cascade systems (not in bi-diagonal form),

• or other specific forms of systems such as for instance symmetric systems for which the matrix operator is symmetric, in this case the matrix operator $\mathcal{M}_2$ becomes

$$\mathcal{M}_2 = \begin{pmatrix} A & C_{21} \\ C_{21}^* & A \end{pmatrix},$$

• and also more general systems with "less structure".

$\rightsquigarrow$ Indeed, systems with less structures are in general more difficult to study.

$\rightsquigarrow$ Cascade systems are in some ways the simplest ones to analyze, and already their analysis requires some work.

So we focus for the moment on cascade systems

- only the last equation is directly controlled,
- they have a cascade structure, i.e. a matrix operator which is triangular, eventually bi-diagonal (when the number of equations is larger or equal to three), when written as second order abstract systems
- and the goal is for all these systems to determine whether if it is possible to drive any initial state to equilibrium for all unknowns $y_1, y_2, \ldots$ by acting only through one control (on the last equation).

This cascade structure appears naturally when:

$\rightsquigarrow$ studying insensitizing controls for scalar equations, that is controls which are robust with respect to a given observation of the solution, when small unknowns perturbations of the initial data may occur.

$\rightsquigarrow$ or for building a simultaneous control for systems coupled in parallel.

The study of under-controlled coupled systems raises several mathematical questions

- Are these systems controllable, i.e. is it possible to "control" the final dynamics to drive the solution to the desired target, acting on less unknowns?
- When the answer is positive, can we determine the optimal geometric conditions both on the control and coupling regions (PDE's formulation) or for a more functional approach (abstract results)?
- Can we characterize the properties of the coupling operators that lead to positive, negative answer to exact controllability?
- How the coupling influence the dynamics of under-controlled systems?
- Can we propose a general methodology to answer, at least partially at a "macroscopic" level, to these questions and draw a picture in a more general context based on intrinsic properties for the cascade systems in abstract form (the wave case being only a peculiar case), and for bounded as well as unbounded control operators?

Some first remarks:

1. The lack of control on the component of the first equation (in y_1) should be compensated by some assumptions on the coupling term $c(.)y_2$ in the first equation. Clearly if $c \equiv 0$, since we have the decoupled system

$$\begin{cases} y_{1,tt} - \Delta y_1 = 0 \text{ in } (0,T) \times \Omega \,, \\ y_{2,tt} - \Delta y_2 = b(x) v \text{ in } (0,T) \times \Omega \,, \end{cases}$$

   the component y_1 cannot be controlled.
2. The discussion on the respective location of $supp\{c\}$ and $supp\{b\}$ is also important. Indeed this is a challenging question and only few results are available in the case $supp\{c\} \cap supp\{b\} = \emptyset$, even for the insensitizing control for parabolic equations which has been widely studied.
3. Can we give sharp conditions on the coupling? This raises the question to determine how the missing "information" can be transferred from the coupling term, and how this information is transported in general geometric conditions.

Some first remarks:

1. The lack of control on the component of the first equation (in y_1) should be compensated by some assumptions on the coupling term $c(.)y_2$ in the first equation. Clearly if $c \equiv 0$, since we have the decoupled system

$$\begin{cases} y_{1,tt} - \Delta y_1 = 0 \text{ in } (0,T) \times \Omega\,, \\ y_{2,tt} - \Delta y_2 = b(x)v \text{ in } (0,T) \times \Omega\,, \end{cases}$$

   the component y_1 cannot be controlled.
2. The discussion on the respective location of $supp\{c\}$ and $supp\{b\}$ is also important. Indeed this is a challenging question and only few results are available in the case $supp\{c\} \cap supp\{b\} = \emptyset$, even for the insensitizing control for parabolic equations which has been widely studied.
3. Can we give sharp conditions on the coupling? This raises the question to determine how the missing "information" can be transferred from the coupling term, and how this information is transported in general geometric conditions.

Some first remarks:

1. The lack of control on the component of the first equation (in y_1) should be compensated by some assumptions on the coupling term $c(.)y_2$ in the first equation. Clearly if $c \equiv 0$, since we have the decoupled system

$$\begin{cases} y_{1,tt} - \Delta y_1 = 0 \text{ in } (0,T) \times \Omega\,, \\ y_{2,tt} - \Delta y_2 = b(x) v \text{ in } (0,T) \times \Omega\,, \end{cases}$$

the component y_1 cannot be controlled.

2. The discussion on the respective location of $supp\{c\}$ and $supp\{b\}$ is also important. Indeed this is a challenging question and only few results are available in the case $supp\{c\} \cap supp\{b\} = \emptyset$, even for the insensitizing control for parabolic equations which has been widely studied.

3. Can we give sharp conditions on the coupling? This raises the question to determine how the missing "information" can be transferred from the coupling term, and how this information is transported in general geometric conditions.

Sommaire

If we deal a single scalar controlled equation, then a wide literature has been developed with different methods to handle the controllability issues:

Multipliers method, micro-analysis approach, Carleman estimates, Fourier decomposition and Ingham types inequalities...
Let us consider a controlled wave equation, that is

$$\begin{cases} y_{tt} - \Delta y = bv \text{ in } (0,T)\times\Omega\,, \\ y = 0 \text{ in } (0,T)\times\Gamma\,, \\ (y, y_t)(0,.) = (y^0, y^1) \text{ in } \Omega\,, \end{cases}$$

for a locally distributed control v and

$$\begin{cases} y_{tt} - \Delta y = 0 \text{ in } (0,T)\times\Omega\,, \\ y = bv \text{ in } (0,T)\times\Gamma\,, \\ (y, y_t)(0,.) = (y^0, y^1) \text{ in } \Omega\,, \end{cases}$$

Here in both cases, b is the control coefficient.

It is in general a nonnegative function which may have a localized support respectively

- in Ω for locally distributed control
- and in Γ for boundary control.

Hence the control is active only within the support of b.

One looks for controls, thus source terms, that drive the solution to a desired target.

It is well-known that by duality arguments, such as the Hilbert Uniqueness Method for instance, the exact controllability is equivalent to

an observability inequality for a dual homogeneous (i.e. without control) system, namely for the system

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } (0, T) \times \Omega\,, \\ u = 0 \text{ in } (0, T) \times \Gamma\,, \\ (u, u_t)(0, .) = (u^0, u^1) \text{ in } \Omega\,, \end{cases}$$

The solution (in the energy space) of the above system is said to satisfy and observability inequality at time T if the following property holds:

$$\exists\ C(T) > 0 \text{ such that for all } (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)\,,$$
$$C(T)||(u^0, u^1)||^2_{H_0^1(\Omega)\times L^2(\Omega))} \leqslant \int_0^T \int_\Omega b|u_t|^2\, dx\, dt\,,$$

in the case of locally distributed control,
and the form

$$\exists\ C(T) > 0 \text{ such that for all } (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)\,,$$
$$C(T)||(u^0, u^1)||^2_{H_0^1(\Omega)\times L^2(\Omega))} \leqslant \int_0^T \int_\Gamma b\left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma\, dt\,,$$

in the case of boundary control.

This can be viewed as a *"quantitative"* unique continuation property, where the unique continuation property is

- in the case of locally distributed observability:

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } (0,T)\times\Omega\,, \\ u = 0 \text{ in } (0,T)\times\Gamma\,, \\ b(.)u_t = 0 \text{ in } (0,T)\times\Omega\,, \rightsquigarrow u_t = 0 \text{ in } (0,T)\times\{x, b(x)\neq 0\}\,, \end{cases}$$

$$\Longrightarrow u = u_t \equiv 0 \text{ in } (0,T)\times\Omega,$$

- and in the case of boundary observability:

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } (0,T)\times\Omega\,, \\ u = 0 \text{ in } (0,T)\times\Gamma\,, \\ b(.)\frac{\partial u}{\partial \nu} = 0 \text{ in } (0,T)\times\Gamma \rightsquigarrow u_t = 0 \text{ in } (0,T)\times\{x, b(x)\neq 0\}\,, \end{cases}$$

$$\Longrightarrow u = u_t \equiv 0 \text{ in } (0,T)\times\Omega.$$

The above observability inequalities hold true only under

- geometric conditions on the support of the control coefficient b,
- for sufficiently large time T (due to the finite speed propagation for the wave equation).

Namely:

• the geometric control condition (GCC) saying that every generalized bi-characteristics traveling at speed 1 should meet the control region at a time $t < T$ (in a non-diffractive point for the boundary case) $\rightsquigarrow$ leads to sharp (sufficient and almost necessary) geometric conditions, requires some smoothness of the control coefficient, the minimal time is geometrically characterized.

• or multiplier type conditions $\rightsquigarrow$ leads to less sharp (explicit) geometric conditions, does not require smoothness of the control coefficient, the time for which the observability estimate holds is not the minimal time in general. Has been extended to Piecewise Multiplier Geometric Conditions (K. Liu 1997).

Sommaire

beamer-tu-logo

beamer-ur-logo

What happens for under-controlled cascade systems in an abstract setting?

Here the main difficulty and challenge is due to the requirement to control a system of two or more coupled equations, by a single (or strictly less than the number of equations) control(s).

Our goals

1. Determine structural properties which allow to answer positively to the question.
2. Identify mathematical properties which indicate how the information is transferred through the coupling.
3. Give sharp results, that is give necessary and sufficient (NS) conditions for such controllability properties to hold. This is linked to the question of optimal geometric conditions on the support of the coupling and control regions.
4. Control more (n) equations by a single control.

- Moreover we want to consider under controlled cascade systems in a more general context than the usual wave equation
- at a macroscopic level, so that we can identify general structural properties
- For this we shall consider abstract controlled cascade systems, for which A is a uniformly elliptic operator (for instance with variable coefficients)
- it can modeled the elasticity operator, plates, beams. . .
- It can also be associated to different types of boundary conditions
- We shall also always consider both bounded (case of locally distributed controls) and unbounded (case of boundary controls) control operators

We consider the abstract dual system (think to wave, Petrowsky, Euler-Bernouilli plates...)

$$\begin{cases} u_1'' + Au_1 = 0\,, \\ u_2'' + Au_2 + C_{21}\,u_1 = 0\,, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2\,, \end{cases}$$

where

- H is an Hilbert space with norm $|\cdot|$ and scalar product $\langle\,,\,\rangle$
- C_{21} is a bounded operator in H
- A satisfies:

$$(A1) \begin{cases} A : D(A) \subset H \mapsto H\,, A^* = A\,, \\ \exists\, \omega > 0\,, |Au| \geq \omega |u| \quad \forall\, u \in D(A)\,, \\ A \text{ has a compact resolvent}\,. \end{cases}$$

Here A can model various mechanical or physical devices.

We want to study the observability properties for this cascade systems by a single observation on the second component in suitable functional spaces.

For this we need some notation:

- We set $H_k = D(A^{k/2})$ for $k \in \mathbb{N}$ and by convention $H_0 = H$ equipped with the corresponding norm and scalar product.
- H_{-k} denotes the dual space of H_k with the pivot space H
- For $V = (v_1, v_2) \in H_k \times H_{k-1}$, we define the energies of level k as

$$e_k(V)(t) = \frac{1}{2}\left(|A^{k/2}v_1|^2 + |A^{(k-1)/2}v_2|^2\right), \ k \in \mathbb{Z}, i = 1, 2.$$

- Whenever w solves $w'' + Aw = 0$ (or $= f$), we set $W = (w, w')$.

• Assume that $\mathbf{B}^*$ satisfies a refined admissibility property for the single forced equation, that is

$$(A_{fa}) \begin{cases} \mathbf{B}^* \in \mathcal{L}(H_2 \times H; G)\,, \forall\, T > 0\ \exists\, C > 0, \\ \text{ such that for all } (w^0, w^1) \in H_1 \times H\,,\ \text{and } f \in L^2([0,T]; H)\,, \\ \text{the solution } w \text{ of } w'' + Aw = f\,, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \leqslant C(e_1(W)(0) + e_1(W)(T) + \\ \int_0^T e_1(W)(t)dt + \int_0^T |f|^2 dt)\,, \end{cases}$$

Then we have

Lemma (A.-B. MCSS 2013, (Admissibility property))

Assume $(A1)$, (A_{fa}), *and* $C_{21} \in \mathcal{L}(H)$, *then for all* $T > 0$, *there exists a constant* $C = C(T) > 0$ *such that for all initial data* U^0, *the solution of the homogeneous dual cascade system satisfies the following direct inequality*

$$\int_0^T ||\mathbf{B}^* U_2||_G^2 \, dt \leqslant C\Big(e_0(U_1)(0) + e_1(U_2)(0)\Big) .$$

Remark

This Lemma establishes a hidden regularity property of the solutions: for all $U^0 \in H \times H_1 \times H_{-1} \times H$, $\mathbf{B}^* U_2 \in L^2([0, T]; G)$.

Let us go back to the dual cascade system

$$\begin{cases} u_1'' + Au_1 = 0\,, \\ u_2'' + Au_2 + C_{21}u_1 = 0\,, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2\,, \end{cases}$$

We want to give necessary and sufficient conditions on the coupling operator C_{21} and on the operator $\mathbf{B}^*$ for the following observability property

$$\int_0^T ||\mathbf{B}^* U_2||_G^2 \geq C(T)\Big(e_0(U_1)(0) + e_1(U_2)(0)\Big)\,,$$

to hold.

Note that the above inequality involves two *different* levels of energies: the natural energy for the directly observed component U_2 and a weakened energy for the unobserved component U_1. This expected form goes back to previous works on symmetric systems (A.-B. 2001, 2003).

Let us go back to the dual cascade system

$$\begin{cases} u_1'' + Au_1 = 0\,, \\ u_2'' + Au_2 + C_{21}u_1 = 0\,, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2\,, \end{cases}$$

We want to give necessary and sufficient conditions on the coupling operator C_{21} and on the operator $\mathbf{B}^*$ for the following observability property

$$\int_0^T ||\mathbf{B}^* U_2||_G^2 \geq C(T)\Big(e_0(U_1)(0) + e_1(U_2)(0)\Big)\,,$$

to hold.

Note that the above inequality involves two *different* levels of energies: the natural energy for the directly observed component U_2 and a weakened energy for the unobserved component U_1. This expected form goes back to previous works on symmetric systems (A.-B. 2001, 2003).

For this we will restrict ourselves to the following class of coupling operators:

• Assume that the coupling operator C_{21} satisfies the following non negativity property (NNP)

$$(NNP)\begin{cases} C_{21}^* \in \mathcal{L}(H)\,, \\ |C_{21}w|^2 \leqslant \beta\langle C_{21}w\,,w\rangle \quad \forall\, w \in H\,, \end{cases}$$

• Note that when $H = L^2(\Omega)$ and $C_{21} = c(\cdot)I$ where c is the coupling coefficient, then the above hypothesis implies that $c \geqslant 0$ a.e. in Ω. Conversely if $c \in L^\infty(\Omega)$ and $c \geqslant 0$ a.e. in Ω, then (NNP) holds. Note that this is a common hypothesis for such types of results (it holds in insensitizing control).

• Note also that at this stage, the null function satisfies (NNP), so that we can already guess that one needs an additional assumption on C_{21} to get an observability estimate for the dual cascade system.

- Assume the following observability inequality for a single equation

$$(A_{obs}) \begin{cases} \exists\, T_0 > 0, \forall\, T > T_0\,, \exists\, C_1(T) > 0 \text{ such that} \\ \forall\, (w^0, w^1) \in H_1 \times H\,, \text{ the solution } w \text{ of} \\ w'' + Aw = 0\,, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \geq C_1(T) e_1(W)(0)\,. \end{cases}$$

- Assume that C_{21} satisfies in addition the following observability property

$$(A_{obsC}) \begin{cases} \exists\, T_0 > 0, \forall\, T > T_0\,, \exists\, C_2(T) > 0 \text{ such that} \\ \forall\, (w^0, w^1) \in H_1 \times H \text{ the solution } w \text{ of} \\ w'' + Aw = 0\,, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T |C_{21} w'|^2 dt \geq C_2(T) e_1(W)(0)\,, \end{cases}$$

Remark

Note that the minimal times for which the two observability inequalities hold in (A_{obs}) and (A_{obsC}) are not necessarily the same for the two observability operators.

Theorem (Sufficient conditions, A.-B. MCSS 2013)

Assume the hypotheses $(A1), (A_{fa}), (A_{obs}), (NNP), (A_{obsC})$*. Then there exists* $T_* > 0$ *such that for all* $T > T_*$*, and all initial data* U^0*, the solution of the dual homogeneous cascade system satisfies the observability estimates*

$$\begin{cases} d_1(T) \int_0^T ||\mathbf{B}^* U_2||_G^2 \geq e_0(U_1)(0)\,, \\ d_2(T) \int_0^T ||\mathbf{B}^* U_2||_G^2 \geq e_1(U_2)(0)\,, \end{cases}$$

where the constants $d_i(T) > 0$ *depend on* T *and satisfy for* T *sufficiently large, suitable asymptotic properties with respect to* T*.*

Note that the above observability inequality is in a *decoupled* form, that is it does not involve the sum of the initial energies of the two components, but each of them separately.

We also prove that the above conditions are optimal in the following theorem.

Theorem (Necessary conditions, A.-B. MCSS 2013)

Assume the hypotheses $(A1)$ *and* (NNP). *Assume that* C_{21} *does not satisfy the observability property given in* (A_{obsC}) *or that* $\mathbf{B}^*$ *does not satisfy* (A_{obs}). *Then there does not exist* $T_* > 0$ *such that for all* $T > T_*$, *the following property holds*

$$(OBS) \begin{cases} \exists\ C > 0 \text{ such that } \forall\ U^0 \text{ the solution satisfies} \\ C(e_0(U_1)(0) + e_1(U_2)(0)) \leqslant C \int_0^T ||\mathbf{B}^* U_2||_G^2\, dt\,. \end{cases}$$

Corollary (A.-B. MCSS 2013)

Assume $(A1)$ *and* (NNP). *Then* (OBS) *holds if and only if* (A_{obsC}) *and* (A_{obs}) *hold.*

To summarize, we prove

Theorem (A.-B. MCSS 2013)

Assume that $\mathbf{B}^*$ is admissible for the "forced" wave equation (assumption (A_{fa})) and that C_{21} satisfies the non negativity property given in (NNP). Then there exists $T_* > 0$ such that for all $T > T_*$, and all initial data $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in H \times H_1 \times H_{-1} \times H$ the solution of the dual cascade system satisfies the observability estimate

$$\int_0^T ||\mathbf{B}^* U_2||_G^2 \geqslant C_1(T)\Big(e_0(U_1)(0) + e_1(U_2)(0)\Big),$$

if and only if

the operator $\mathbf{B}^*$ satisfies an observability inequality for all $T \geq T_0$ for a single scalar abstract equation (property (A_{obs}))

and C_{21} satisfies the observability inequality (A_{obsC}).

Hence we give a necessary and sufficient condition in the class of bounded coupling operators C_{21} satisfying (NNP) (or equivalently a non positivity property).

The proof is based on

- the two-level energy method (A.-B. 2001, 2003) introduced for systems coupled symmetrically and coercive couplings
- its extension to handle partially coercive couplings (A.-B. and Léautaud 2011, 2012) for symmetrically coupled systems

The spirit of the proof is to:

compensate the lack of observation of the second component by a balance effect between the natural energy of the observed component and the weakened energy of the unobserved one.

beamer-tu-lo

beamer-ur-logo

Hence we give a necessary and sufficient condition in the class of bounded coupling operators C_{21} satisfying (NNP) (or equivalently a non positivity property).

The proof is based on

- the two-level energy method (A.-B. 2001, 2003) introduced for systems coupled symmetrically and coercive couplings
- its extension to handle partially coercive couplings (A.-B. and Léautaud 2011, 2012) for symmetrically coupled systems

The spirit of the proof is to:

compensate the lack of observation of the second component by a balance effect between the natural energy of the observed component and the weakened energy of the unobserved one.

The ingredients for the two-level energy method are:

1. a key estimate due to (*NPP*) for the coupling $+$ the observability hypothesis on the coupling.
2. observability assumption for a forcing source term, uniform with respect to sufficiently large times T. This is a key property introduced already in A.-B. 2003 but proved in applications to PDE's by multipliers methods, and generalized later on by A.-B. and Léautaud in JMPA 2012 for the abstract forced wave equation (**in a form invariant by time-translation**). This is this property which allowed us to handle situations for which the coupling and control regions do not meet (when applied to given realizations of A).
3. energy type estimates (several ones are required).
4. conservation of the total natural and weakened energies and suitable balance of energies in the case of symmetrically coupled systems.
5. this property is lost for cascade systems, however we proved that the two-level energy method can be extended to handle this case.

Why invariance properties such as conservation of energy are important?

The usual way to prove a boundary observability inequality for the second equation (indeed for the scalar wave equation) in case of a vanishing u_1 by multiplier methods is to perform integration by parts for the term

$$I = \int_0^T \int_\Omega \Big(u_{2,tt} - \Delta u_2\Big)(m \cdot \nabla u_2 + u_2(n-1)/2)\, dxdt\,.$$

for strong solutions, where $m(x) = x - x_0$. In this case, one has to replace (GCC) by a stronger multiplier condition, namely

$$m \cdot \nu > 0 \text{ on } \Gamma_1 \text{ where } b > 0 \text{ on } \overline{\Gamma}_1\,,\, m \cdot \nu \leq 0 \text{ on } \Gamma \backslash \Gamma_1\,.$$

These integrations by parts allow to rewrite I as

$$I = \int_0^T \int_\Omega \left(|u_{2,t}|^2 + |\nabla u_2|^2 \right) + \Big[\int_\Omega u_{2,t}(m \cdot \nabla u_2 + u_2(n-1)/2) \Big]_0^T - \frac{1}{2} \int_0^T \int_\Gamma m \cdot \nu \Big| \frac{\partial u_2}{\partial \nu} \Big|^2 d\sigma dt ,$$

so that since $u_{2,tt} - \Delta u_2 + c(x) u_1 = 0$, we have

$$\frac{1}{2} \int_0^T \int_{\Gamma_1} m \cdot \nu \Big| \frac{\partial u_2}{\partial \nu} \Big|^2 d\sigma dt = -\frac{1}{2} \int_0^T \int_{\Gamma \setminus \Gamma_1} m \cdot \nu \Big| \frac{\partial u_2}{\partial \nu} \Big|^2 d\sigma dt + \int_0^T e(u_2)\, dt + \Big[\int_\Omega u_{2,t}(m \cdot \nabla u_2 + u_2(n-1)/2) \Big]_0^T + \int_0^T \int_\Omega c(x) u_1 (m \cdot \nabla u_2 + u_2(n-1)/2)\, dx\, dt .$$

where $e(u_2)$ stands for the energy of u_2.

Thanks to our geometric assumptions, we have

$$\frac{1}{2}\int_0^T\int_{\Gamma_1} m\cdot\nu\left|\frac{\partial u_2}{\partial\nu}\right|^2 d\sigma dt \leq \frac{R}{2b_-}\int_0^T\int_{\Gamma_1} b\left|\frac{\partial u_2}{\partial\nu}\right|^2 d\sigma dt,$$

where $b \geq b_- > 0$ on Γ_1. The right side can be bounded below by

$$\int_0^T e(u_2)\,dt - C(e(u_2)(T) + e(u_2)(0)) +$$

$$\int_0^T\int_\Omega c(x)u_1(m\cdot\nabla u_2 + u_2(n-1)/2)\,dx\,dt \rightsquigarrow$$

additional term due to the coupling effect .

If u_1 vanishes, the energy of u_2 is conserved.

This is a key point in this type of method.

We use this property twice: to rewrite

$$\int_0^T e(u_2)\,dt = Te_2(u_2)(0)\,,$$

and

$$e(u_2)(T) = e_2(u_2)(0)\,.$$

so that we then easily get that

$$\frac{R}{2b_-}\int_0^T \int_{\Gamma_1} \left|\frac{\partial u_2}{\partial \nu}\right|^2 d\,\sigma d\,t \geq (T - 2C)e(u_2)(0)\,,$$

which is the usual boundary observability inequality . $\rightsquigarrow$ Note that that is the reason why the observability estimates is proved for T large enough

Hence to prove the desired observability inequalities when u_1 is non vanishing, one has to find out how to handle the coupling term cu_1 in the above estimate.

The "natural" way to get an estimate for the forced wave equation when c is non vanishing is to use Cauchy-Schwarz inequality to handle the forced (coupling term).

This leads to

$\exists\ \eta_i > 0$ for $i = 0, \ldots, 3$ such that for all $T > T_0$,

$$\eta_0 \int_0^T \int_{\Gamma_1} \left|\frac{\partial u_2}{\partial \nu}\right|^2 d\,\sigma d\,t\,dt \geqslant (1 - \eta_1 \delta) \int_0^T e(u_2)(t)\,dt$$
$$-\eta_2(e(u_2)(0) + e(u_2)(T)) - \eta_3 \delta^{-1} \int_0^T |cu_1|^2\,dt\,, \forall\ \delta \in (0, \eta_1^{-1}).$$

beamer-tu-logo

beamer-ur-logo

This property can be proved form an abstract point of view and in a form which is invariant by time-translation $\rightsquigarrow$ item 2:

Lemma (A.-B. & Léautaud, JMPA 2012, Uniform observability property for a forced scalar equation)

We assume the hypotheses $(A1)$, $(A)_{fa}$ and (A_{obs}). Then, there exist constants $\eta_0 > 0$ and $\alpha_0 > 0$ such that for all $T > T_0$, and for any solution $P = (p, p')$ of the nonhomogeneous equation

$$p'' + Ap = f \in L^2([0,T]; H), \tag{1}$$

the following uniform observability estimate holds

$$\eta_0 \int_0^T ||\mathbf{B}^* P||^2 \, dt \geqslant \int_0^T e_1(P)(t) \, dt - \alpha_0 \int_0^T |f|^2 \, dt .$$

This is important to get optimal geometric conditions on coupling and control regions when applied to examples of PDE's.

Hence the coupling term perturbs the property of conservation of energy. At least for cascade system of two equations, it perturbs the second equation (not the first one).

$\rightsquigarrow$ So in this aspect, it bothers us and we want to "get rid" of it

$\rightsquigarrow$ On the other hand, if the coupling is not there, it does not work!

• So we need to find a way to capture the good influence of the coupling and then to "get rid" of it at the right places in the estimates.

$\rightsquigarrow$ This good influence is generated by some non negativity property and some observability property $\rightsquigarrow$ some partial coercivity property of the coupling.

$\rightsquigarrow$ get rid of its influence on the conservation properties, taking care of the dependence with respect to "large T". This is done through suitable energy estimates.

The argument is much more tricky for more coupled systems than cascade systems, such as symmetrically coupled system: one further needs to find a right balance of the total weakened and natural energies developed in A.-B. 2003.

Concerning methods:

Here the positive controllability/observability results are based on the zero order terms due to the coupling and their coercivity properties. This is a main point in the two-level energy method.

The way to "measure" this positive effect is to work in a bigger space $H \times H_{-1}$ for the unobserved component. That is why it is important to work with *two (or more as we will see later on) different levels of energies*.

Note that the methods based on micro-local analysis or Carleman estimates, neglect in general lower order terms, since they are absorbed by dominant terms (high frequencies or large values of parameters) $\rightsquigarrow$ can be handled thanks to the use of different levels of energies.

The two-level energy method is constructive so that it allows us to obtain quantitative results.

- Another important point is to understand that the coupling operator can be localized, in this case the information contained in the coupling "propagates" provided that the coupling region satisfies (GCC).

- This has first been noted in my work in collaboration with Léautaud (CRAS 2012, JMPA 2013) on symmetrically coupled systems with localized couplings.

- Note that the (NS) condition is written in a way that does not depend on the mathematical technique to prove observability estimates for the scalar abstract equation, so that one may combine different techniques to relax for instance the smoothness assumptions, or to have the sharpest geometric conditions, . . .

Positive results on cascade or symmetric systems as said before in:

- A.-B. SICON 2003 for symmetrically coupled systems of two abstract wave equations in case of coercive coupling operators, with different diffusion operators.

- A.-B. and Léautaud CRAS 2011, JMPA 2012 for symmetrically coupled systems of two abstract wave equations in case of partially coercive couplings and the same diffusion operators.

For specific systems, namely the wave equation:

• Rosier and de Teresa 2011 for a one dimensional system of two $1-D$ wave equations coupled in cascade. It is a constructive proof. It does not require smoothness of the coupling (it is a characteristic function). The result is based on Dàger's 1-D approach (periodicity of the solution of the $1-D$ free wave equation).

• Dehman Le Rousseau and Léautaud (2013) for the case of 2-coupled cascade systems in a $\mathcal{C}^\infty$ compact connected riemannian manifold without boundary and for locally distributed control with an implicit geometric characterization of the minimal control time via micro-local analysis.

$\rightsquigarrow$ *proof is by contradiction, based on micro-local analysis and also on the idea to work in the weakened energy spaces for the unobserved component from the two-level energy method (otherwise these types of approach does not "see" the coercivity properties of the coupling operator).*

For specific systems, namely the wave equation:

• Rosier and de Teresa 2011 for a one dimensional system of two $1-D$ wave equations coupled in cascade. It is a constructive proof. It does not require smoothness of the coupling (it is a characteristic function). The result is based on Dàger's 1-D approach (periodicity of the solution of the $1-D$ free wave equation).

• Dehman Le Rousseau and Léautaud (2013) for the case of 2-coupled cascade systems in a $\mathcal{C}^\infty$ compact connected riemannian manifold without boundary and for locally distributed control with an implicit geometric characterization of the minimal control time via micro-local analysis.

$\rightsquigarrow$ *proof is by contradiction, based on micro-local analysis and also on the idea to work in the weakened energy spaces for the unobserved component from the two-level energy method (otherwise these types of approach does not "see" the coercivity properties of the coupling operator).*

Indeed we can observe that

Remark

- *Symmetric systems are "more coupled" than cascade systems:*

  Indeed when the initial data for U_1 is vanishing in the dual cascade system, then $U_1 \equiv 0$ for all times so that the system reduces to a scalar wave equation. This property does not hold true for symmetric systems. This also tells than the study of cascade systems is somehow easier *than the study of symmetric (or even more general coupled systems).*

- Decoupled *versus* Coupled *observability estimates:*

  It also explains why we can obtain decoupled *observability inequalities for the dual cascade system, and respectively a* coupled *observability inequality for the dual symmetric system.*

Applications:

In the 1-D cascade system of two wave equations, namely:

$$\begin{cases} u_{1,tt} - u_{1,xx} = 0\,, \quad (t,x) \in (0,T) \times (0,1)\,, \\ u_{2,tt} - u_{2,xx} + c(.)u_1 = 0\,, \quad (t,x) \in (0,T) \times (0,1)\,, \\ u_i = 0\,,\ \text{in } (0,T) \times \{0,1\} \text{ for } i = 1,2\,, \\ (u_i, u_i')(0,.) = (u_i^0, u_i^1)(.) \quad \text{in } (0,1)\,,\ \text{for } i = 1,2\,, \end{cases}$$

we obtain a positive observability estimate for coupling coefficients such that $c \geqslant 0$ a.e. on $(0,1)$, **if and only if**

- $c \geqslant c_- > 0$ on any non empty open subset O of $(0,1)$ and the control coefficient $b \geqslant 0$ a.e. on $(0,1)$ (or on its boundary)
- and $b \geqslant b_- > 0$ on any open non-empty subset ω of Ω (or at 0 or 1 in the case of boundary observability).

These results extend to multi-D case:

• either using (GCC) on both the control and coupling regions, allowing situations for which both regions do not meet, but then one needs some smoothness of these coefficients,

• or using the Multiplier type Geometric Conditions for the control and coupling regions (it includes the cases for which c and b are characteristic functions of some suitable sets).

• for wave cascade systems, plates cascade systems, ... The set Ω can be a smooth connected compact Riemannian manifold with or without boundary, A can be given by $A = -\Delta_c = -\text{div}(d\nabla)$ a (positive) elliptic operator (or the Laplace Beltrami operator with respect to the Riemannian metric) on Ω

• for locally distributed as well as boundary observation operators.

$\rightsquigarrow$ does not characterize the minimal control time.

$\rightsquigarrow$ these results allowed us to solve the insensitizing question raised by J.-L. Lions in a multi-dimensional setting and general geometric conditions (A.-B. MCSS 2013).

Sommaire

We consider the dual bi-diagonal cascade systems of size n

$$\begin{cases} u_1'' + Au_1 = 0\,, \\ u_i'' + Au_i + C_{ii-1}u_{i-1} = 0\,, 2 \leqslant i \leqslant n\,, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, \dots n\,, \end{cases} \iff \begin{cases} U' = \mathcal{A}_n U\,, \\ U(0) = U^0 \end{cases}$$

where $U = (u_1, \dots, u_n, u_1', \dots, u_n')$.

In view of applications to simultaneous control of certain systems, we would like to determine sufficient (eventually also necessary) conditions, so that observing only the last component, we can recover the energy of initial data for all the components.

Hence the purpose is to generalize the previous results for cascade systems of 2 equations to bi-diagonal cascade systems of n equations.

We shall assume that the coupling operators C_{ii-1} for $i = 2, \dots, n$ satisfy

$$(A2)_n \begin{cases} \forall\, i \in \{2, \dots, n\} : \\ C^*_{ii-1} \in \mathcal{L}(H_k)\ \forall\, k \in \{0, \dots n-i+1\}\,, \\ |C_{ii-1} w|^2 \leqslant \beta_i \langle C_{ii-1} w\,, w\rangle\ \forall\, w \in H\,, \end{cases}$$

and

$$(A3)_n \begin{cases} \text{For all } i \in \{2, \dots, n\},\ \ \exists\, T_{0,i} > 0, \forall\, T > T_{0,i}, \exists\, \gamma_i(T) > 0, \\ \text{such that all the solutions } w \text{ of } w'' + Aw = 0 \text{ satisfy} \\ \int_0^T |C_{ii-1} w'|^2 dt \geqslant \gamma_i(T) e_1(W)(0)\,. \end{cases}$$

For a given $i \in \{2, \dots, n\}$, we denote by G_i given Hilbert spaces with norm $||\ ||_{G_i}$ and scalar product $\langle\, ,\rangle_{G_i}$. The spaces G_i, $i = 2, \dots, n$ will be identified to their dual spaces in all the sequel. Let $\mathbf{B}_\mathbf{n}^*$ for $n \geq 2$ be an arbitrary observability operator satisfying the following assumptions:

$$(A4)_n \begin{cases} \mathbf{B}_\mathbf{n}^* \in \mathcal{L}(H_2 \times H; G_n), \\ \forall\ T > 0 \ \exists\ D_n = D_n(T) > 0, \text{ such that all the solutions } w \text{ of} \\ w'' + Aw = f \in L^2([0,T]; H) \text{ satisfy} \\ \int_0^T \|\mathbf{B}_\mathbf{n}^*(w, w')\|_{G_n}^2 dt \leqslant D_n(T)\Big(e_1(W)(0) + e_1(W)(T) + \\ \int_0^T e_1(W)(t)dt + \int_0^T |f|^2 dt\Big), \end{cases}$$

where $W = (w, w')$.

$$(A5)_n \begin{cases} \exists\ T_{0,n} > 0, \forall\ T > T_{0,n}, \exists\ R_n(T) > 0 \\ \text{such that all the solutions } w \text{ of } w'' + Aw = 0 \text{ satisfy} \\ \int_0^T \|\mathbf{B}_\mathbf{n}^*(w, w')\|_{G_n}^2 dt \geqslant R_n(T) e_1(W)(0), \end{cases}$$

Theorem (A.-B. 2013, Advances in Differential Equations)

Assume that the observation operator $\mathbf{B}_\mathbf{n}^*$ *satisfies* $(A4)_n$ *and that the coupling operators on the sub-diagonal satisfy* $(A2)_n$ *(smoothness and partial coercivity properties as seen for* 2*-coupled cascade systems)*

Assume $(A1)$, $(A4)_n$ *and* $(A2)_n$ *for all* $i \in \{2, \ldots, n\}$. *Then there exists* $T_n^* > 0$ *such that for all* $T > T_n^*$

$$(OBS)_n \begin{cases} \exists\ C > 0 \text{ such that } \forall\ U^0 \text{ the solution satisfies} \\ C \displaystyle\sum_{i=1}^{n} e_{1+i-n}(U_i)(0) \leqslant \int_0^T \|\mathbf{B}_\mathbf{n}^* U_n\|_{G_n}^2 \, dt\,. \end{cases}$$

holds if and only if $(A3)_n$ *and* $(A5)_n$ *hold. Moreover,* T_n^* *has to be greater than* $\max(\max_{2 \leq i \leqslant n}(T_{i,c}), T_{0,n})$ *where* $T_{i,c}$ *for* $i = 2, \ldots, n$ *denote for each observability operator* $\Pi_i = C_{ii-1}$ *the minimal control times for which* $(A3)_n$ *holds, and* $T_{0,n}$ *the minimal control time for which* $(A5)_n$ *holds.*

The proof is a generalization of the proof for $n = 2$ (2-coupled cascade systems).

$\rightsquigarrow$ It relies on a tricky induction argument and requires a careful analysis of how the lack of observation of the $n - 1$ first components can be compensated by the coupling terms. The induction argument invokes several estimates with suitable asymptotic estimates for large times.

$\rightsquigarrow$ The multi-levels energy method is constructive. It uses the property that one can derive from the original system set in the natural energy space a hierarchy of related systems similar to the original one, but set in weakened energy spaces.

$\rightsquigarrow$The solutions of these hierarchic systems are linked to each other, and this rich structure allows us to get positive controllability results.

$\rightsquigarrow$ The subclass of cascade bi-diagonal system can be seen as a toy model.

Let $n \geq 2$ be given. The proof of the above result relies on the following property at order n:

$$(\mathcal{P}_n) \begin{cases} \exists\, K_n > 0\,, \exists\, T_n^* > 0\,,\ \text{s. t.}\ \forall\, T > T_n^*,\ \exists\, r_{n,n}(T) > 0\,, \\ \forall\, i \in \{1, \ldots, n\},\ \exists\, d_{i,n}(T) > 0, k_{i,n}(T) > 0 \text{ s. t. for all solutions} \\ \ \text{of } \mathbf{W}' = \mathcal{A}_{\mathbf{n}} \mathbf{W}\ : \\ e_{1+i-n}(W_i)(0) \leq d_{i,n}(T) \int_0^T ||\mathbf{B}_{\mathbf{n}}^*(W_n)||_{G_n}^2 dt \quad \forall\, i \in \{1, \ldots, n\}\,, \\ e_0(W_{n-1})(T) \leq d_{n-1,n}(T) \int_0^T ||\mathbf{B}_{\mathbf{n}}^*(W_n)||_{G_n}^2 dt\,, \\ \int_0^T \langle C_{nn-1} w_{n-1}, w_{n-1} \rangle \, dt \leq r_{n,n}(T) \int_0^T ||\mathbf{B}_{\mathbf{n}}^*(W_n)||_{G_n}^2 dt\,, \\ \int_0^T e_{1+i-n}(W_i)(t)\, dt \leq k_{i,n}(T) \int_0^T ||\mathbf{B}_{\mathbf{n}}^*(W_n)||_{G_n}^2 dt\ \forall\, i \in \{1, \ldots, n\}\,, \end{cases}$$

where the dependence of the constants $d_{i,n}(T), r_{n,n}(T), k_{i,n}(T)$ for $i \in \{1, \ldots, n\}$ with respect to T is important (and is part of the proof).

Theorem (A.-B. ADE 2013)

Let $n \geq 2$ be an integer. We assume that for all $i \in \{2, \ldots, n+1\}$,

- *the operators C_{ii-1} satisfy the assumption $(A2)_{n+1} - (A3)_{n+1}$.*
- *We assume that for all $k \in \{2, \ldots, n\}$ the property $(\mathcal{P}_k)$ holds for any observation operator $\mathbf{B_k^*}$, satisfying $(A4)_k - (A5)_k$.*

Then the property $(\mathcal{P}_{n+1})$ also holds for any observation operator $\mathbf{B_{n+1}^*}$ satisfying $(A4)_{n+1} - (A5)_{n+1}$.

$\rightsquigarrow$ The "price" to pay is that going from the last equation towards the first one, we can reconstruct the initial data of the corresponding component, but in weaker and weaker energy spaces, namely in $H_{1-n+i} \times H_{i-n}$ for the component $U_i = (u_i, u_i')$ (linked to the domains of fractional powers of A).

$\rightsquigarrow$ Namely in $H_{1-n+i} \times H_{i-n}$ for the component $U_i = (u_i, u_i')$. Hence, the involved energy of U_i becomes weaker as i goes away from n which is the rank of the observed component.

$\rightsquigarrow$ The regularity hypotheses on the coupling operators on the sub-diagonal may imply compatibility conditions for a sufficiently large number of equations. For $n \leqslant 5$, it is still possible to have control and coupling regions which do not meet.

Sommaire

beamer-tu-lo

beamer-ur-logo

We recall that within the class of coupling operators C satisfying

$$(NNP)\begin{cases} C_{21}^* \in \mathcal{L}(H)\,, \\ |C_{21}w|^2 \leqslant \beta \langle C_{21}w\,, w\rangle \quad \forall\, w \in H\,, \end{cases}$$

we give above a general necessary and sufficient condition on C_{21} and on the observation operator, valid for general abstract cascade systems (valid for instance for wave as well as plate equations,...), and for locally distributed as well as boundary controlled systems.

- What happens if we relax this property?

- In particular if $H = L^2(\Omega)$ and $C_{21} = c(.)\, I$, when the coupling coefficient $c \in L^\infty(\Omega)$ changes sign in Ω?

We recall that within the class of coupling operators C satisfying

$$(NNP)\begin{cases} C_{21}^{*} \in \mathcal{L}(H)\,, \\ |C_{21}w|^2 \leqslant \beta\langle C_{21}w\,,w\rangle \quad \forall\, w \in H\,, \end{cases}$$

we give above a general necessary and sufficient condition on C_{21} and on the observation operator, valid for general abstract cascade systems (valid for instance for wave as well as plate equations,...), and for locally distributed as well as boundary controlled systems.

- What happens if we relax this property?

- In particular if $H = L^2(\Omega)$ and $C_{21} = c(.)\,I$, when the coupling coefficient $c \in L^{\infty}(\Omega)$ changes sign in Ω?

beamer-tu-logo

beamer-ur-logo

We recall that within the class of coupling operators C satisfying

$$(NNP)\begin{cases} C_{21}^{*} \in \mathcal{L}(H)\,, \\ |C_{21}w|^2 \leqslant \beta\langle C_{21}w\,, w\rangle \quad \forall\, w \in H\,, \end{cases}$$

we give above a general necessary and sufficient condition on C_{21} and on the observation operator, valid for general abstract cascade systems (valid for instance for wave as well as plate equations,...), and for locally distributed as well as boundary controlled systems.

- What happens if we relax this property?

- In particular if $H = L^2(\Omega)$ and $C_{21} = c(.)\,I$, when the coupling coefficient $c \in L^\infty(\Omega)$ changes sign in Ω?

beamer-tu-logo

beamer-ur-logo

Theorem (A.-B. 2013)

Assume $(A1)$ *and that* $\mathbf{B}^*$ *satisfies* $(A)_{fa}$, *and* $(A)_{obs}$. *Then, there exist operators* $C_{21} \in \mathcal{L}(H)$ *which are not satisfying the non negativity property (nor a NPP) but for which the solutions of the dual cascade system*

$$\begin{cases} u_1'' + Au_1 = 0\,, \\ u_2'' + Au_2 + C_{21}u_1 = 0\,, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2\,, \end{cases}$$

satisfy the observability inequality

$$\int_0^T ||\mathbf{B}^* U_2||_G^2 \geqslant C(T)\Big(e_0(U_1)(0) + e_1(U_2)(0)\Big)\,,$$

Remark

This includes in particular examples for which $H = L^2(\Omega)$, $C_{21} = c(\cdot)I$ *and for which* c *changes sign in* Ω.

Theorem (A.-B. 2013)

There exist examples of coupling coefficients

- *which are changing sign within Ω and,*
- *for which we can exhibit infinite dimensional subspaces of initial data for which the dual cascade system does not satisfy the unique continuation property,*
- *for which, for certain initial data, weaker observability inequalities hold*

Similar examples of non unique continuation results can be built for parabolic cases as well.

This shows that the situation is much richer when we relax the hypothesis (*NNP*), that is when the coupling coefficient c changes sign within Ω.

- unique continuation properties can be true or not,
- observability properties can be true or not, or hold in weaker spaces than expected,
- controllability properties and thus the possibility to drive the dynamics of under-controlled systems can hold or not

so that all these properties highly depend on the properties of the coupling operator.

Sommaire

We are interested in capturing essential features of (at least some) under-controlled systems, at a general level.

If we want to have a clear understanding of under-controlled systems, one has to "identify" classes of systems having some common mathematical and intrinsic properties and to be able to give negative results.

We give a powerful and flexible method which allows to give general results on several classes of under-controlled systems :

- Cascade systems of 2 equations
- Symmetric systems of 2 equations
- Higher order cascade systems (bi-diagonal, mixed ones)
- Multi-D, locally distributed as well as boundary controls

and allow us to obtain sharp and general results on applications to insensitizing controls and some tracks to simultaneous systems coupled in parallel.

We are interested in capturing essential features of (at least some) under-controlled systems, at a general level.

If we want to have a clear understanding of under-controlled systems, one has to "identify" classes of systems having some common mathematical and intrinsic properties and to be able to give negative results.

We give a powerful and flexible method which allows to give general results on several classes of under-controlled systems :

- Cascade systems of 2 equations
- Symmetric systems of 2 equations
- Higher order cascade systems (bi-diagonal, mixed ones)
- Multi-D, locally distributed as well as boundary controls

and allow us to obtain sharp and general results on applications to insensitizing controls and some tracks to simultaneous systems coupled in parallel.

beamer-tu-logo

beamer-ur-logo

We are interested in capturing essential features of (at least some) under-controlled systems, at a general level.

If we want to have a clear understanding of under-controlled systems, one has to "identify" classes of systems having some common mathematical and intrinsic properties and to be able to give negative results.

We give a powerful and flexible method which allows to give general results on several classes of under-controlled systems :

- Cascade systems of 2 equations
- Symmetric systems of 2 equations
- Higher order cascade systems (bi-diagonal, mixed ones)
- Multi-D, locally distributed as well as boundary controls

and allow us to obtain sharp and general results on applications to insensitizing controls and some tracks to simultaneous systems coupled in parallel.

We are interested in capturing essential features of (at least some) under-controlled systems, at a general level.

If we want to have a clear understanding of under-controlled systems, one has to "identify" classes of systems having some common mathematical and intrinsic properties and to be able to give negative results.

We give a powerful and flexible method which allows to give general results on several classes of under-controlled systems :

- Cascade systems of 2 equations
- Symmetric systems of 2 equations
- Higher order cascade systems (bi-diagonal, mixed ones)
- Multi-D, locally distributed as well as boundary controls

and allow us to obtain sharp and general results on applications to insensitizing controls and some tracks to simultaneous systems coupled in parallel.

This method is based on energy and invariance properties such as

- invariance with time translations,
- conservation (time conservation of the energy)
- change of sense of time (irreversibility) (for insensitizing control)

We can generalize these results (works in progress) in several directions

but still a lot remains to be done for a clear and large understanding of these systems and of their structural properties.

- **Unstructured systems :** coupling matrix operators which do not have a lower (or upper) triangular, or symmetric structure.
- **Space and time dependent coupling operators :** in this case, we loose invariance with respect to time translations.
- **Different dynamics :** for instance different diffusion operators, mixing of hyperbolic and parabolic equations, . . .

⋮

Thanks for your attention